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The geometry of uniserial representations of finite dimensional algebras I[1]

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Abstract

It is shown that, given any finite dimensional, split basic algebra $\Lambda = K\Gamma/I$ (where Γ is a quiver and I an admissible ideal in the path algebra $K\Gamma$), there is a finite list of affine algebraic varieties, the points of which correspond in a natural fashion to the isomorphism types of uniserial left Λ-modules, and the geometry of which faithfully reflects the constraints met in constructing such modules. A constructive coordinatized access to these varieties is given, as well as to the accompanying natural surjections from the varieties onto families of uniserial modules with fixed composition series. The fibres of these maps are explored, one of the results being a simple algorithm to resolve the isomorphism problem for uniserial modules. Moreover, new invariants measuring the complexity of the uniserial representation theory are derived from the geometric viewpoint. Finally, it is proved that each affine algebraic variety arises as a variety of uniserial modules over a suitable finite dimensional algebra, in a setting where the points are in one-one correspondence with the isomorphism classes of uniserial modules. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

This is the first part of a trilogy which is to lay the foundations for a geometric approach to the uniserial representations of finite dimensional algebras. The need for a solid understanding of the full class of uniserials arose within a program aimed at approximating finitely generated modules by modules of a simpler structure, the

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basic building blocks of 'helpful' approximations being uniserial modules (cf. [6]). Of course, a uniserial module viewed by itself does not hold much interest, as is the case for a point on a curve when considered outside the context of the curve. In classifying families of uniserial modules, the interest lies in the number of and interplay among the parameters which offer themselves for the description of these modules.

For good focus, let us start by recording two problems which have been propagated by Maurice Auslander since the 1970s and have now appeared among the eleven open problems stated in [2, pp. 411–412]:

(1) Give a method for deciding when two uniserial modules over an artin algebra are isomorphic.

(2) Which artin algebras of infinite representation type have only a finite number of pairwise nonisomorphic uniserial modules?

Specializing to finite dimensional algebras over algebraically closed fields, we present a solution to problem 1 and supply the tools to tackle question 2 in [8]. More generally, our results address split basic finite dimensional algebras Λ over an arbitrary field K, i.e., algebras of the form $K\Gamma/I$, where Γ is a quiver and I an admissible ideal in the path algebra $K\Gamma$. Given such a 'coordinatization' of Λ, we introduce irreducible affine algebraic varieties over K, the points of which parametrize the isomorphism types of uniserial Λ-modules, and the geometry of which reflects the constraints met in constructing such modules. While being a priori defined in terms of quiver and relations of Λ, this list of varieties will turn out to be uniquely determined by the isomorphism type of Λ, up to order and birational equivalence [5].

There are several 'classical' ways of viewing representations as points of algebraic varieties; all of these varieties contain classes of uniserial representations as open subvarieties. The most time-honored procedure is to fix a K-basis for Λ, say $\lambda_1, \ldots, \lambda_m$, and to define the variety $\mathbb{M}\mathrm{od}_d(\Lambda)$ of d-dimensional Λ-modules as a closed subvariety of $M_d(K)^m$ where the matrix in the ith slot represents module multiplication by λ_i. The sets of points corresponding to the isomorphism classes of modules arise as orbits under the obvious $GL_d(K)$-action (cf. [9, Sections 12.16, 12.17]). It is not difficult to see that the d-dimensional uniserial modules with a fixed sequence of consecutive composition factors form an open subvariety of $\mathbb{M}\mathrm{od}_d(\Lambda)$, which is invariant under the canonical $GL_d(K)$-action. However, these varieties are very large, and the information they contain is encoded in the GL-actions. In contrast, the ones which we will introduce and study here provide a close fit for the uniserial modules. In many cases, they are isomorphic to the geometric quotient of the above-mentioned uniserial subvarieties of $\mathbb{M}\mathrm{od}_d(\Lambda)$ modulo the GL-action; in particular, there is then a 1–1 correspondence between points and isomorphism types. When there is not, the 'slack' occurring in the varieties turns out to be fairly harmless.

For more detail, let us concentrate on the class of uniserial representations of length $l+1$ having a fixed sequence $\mathbb{S} = (S(1), \ldots, S(l+1))$ of simple composition factors. It turns out that there is a natural subdivision of this class, possibly with overlaps, so that each of the segments is described by an *irreducible* affine variety. A primary, somewhat rougher, subdivision is in terms of 'masts', as follows. Clearly, for each uniserial

Λ-module U with sequence $\mathbb{S}$ of consecutive composition factors, there exists at least one path p of length l in $K\Gamma$ such that $pU \neq 0$; necessarily p passes in order through the sequence $(e(1), \dots, e(l+1))$ of those vertices in Γ which represent the simple modules $S(i)$. (Here we consider U as a $K\Gamma$-module in the obvious fashion.) Each such path will be called a *mast* for U, and for each mast p we will construct an affine variety V_p over K – not necessarily irreducible – and a canonical surjection Φ_p from V_p onto the set of isomorphism types of uniserial left Λ-modules with mast p (Section 3). If Γ does not have any double arrows, there is clearly at most one path p passing through the sequence $(e(1), \dots, e(l+1))$ of vertices, and the variety V_p parametrizes the full set of isomorphism classes of uniserial modules with composition sequence $\mathbb{S}$. In case Γ does contain double arrows, several varieties of the form V_p are needed to account for the uniserial modules with composition sequence $\mathbb{S}$ in general. In this case, the irreducible components of all the pertinent V_p's are combined into a family $V_{\mathbb{S}}$ of varieties, as discussed below. Nonetheless, the 'packaging' of irreducible components in terms of masts remains the most accessible for both proofs and computations and hence will play a crucial role in our work.

Loosely speaking, the points of V_p are – as in the classical approach – strings of coordinate vectors determining module multiplication. But the bases for the $(l+1)$-dimensional K-space underlying the representations in the image of Φ_p used here are being shifted from one point of V_p to the next, in a fashion that is tied to the multiplication of the uniserial modules labeled by these points. While we lose the natural GL_{l+1}-action coming with $\mathbb{M}\mathrm{od}_{l+1}(\Lambda)$ in the process, we gain, among other things, a particularly transparent connection between the points of V_p and the graphs of the uniserial modules they represent, as well as a geometric picture which cleanly shows how the relations in the ideal I impinge on the interplay of the parameters of the uniserials. As another bonus of restricting our focus to uniserial representations, we can make do with a small selection of paths from $K\Gamma$ as representative 'multipliers' and with bases for small K-subspaces of the modules considered in order to pin down the effect of multiplication. The main price we pay for tight fit and manageability, on the other hand, lies in the fact that independence of the chosen coordinate system for Λ is not at all self-evident in this setting. For instance, certain irreducible components of V_p may shift to a different variety V_q under a change of coordinatization. However, if we denote by $V_{\mathbb{S}}$ the full collection of all the irreducible components of the varieties V_q, where q runs through the paths of length l that pass through the sequence $(e(1), \dots, e(l+1))$, the isomorphism type of Λ uniquely determines $V_{\mathbb{S}}$ up to birational equivalence. In case Γ has no double arrows, the irreducible components in $V_{\mathbb{S}}$ are even unique up to isomorphism. Uniqueness will be proved in a subsequent joint note with Bongartz [5]. It turns out that translating the varieties considered here into certain closed subvarieties of $\mathbb{M}\mathrm{od}_{l+1}(\Lambda)$ provides the most convenient setting for this purpose.

To understand the emerging picture, we further require an in-depth study of intersections of the form $\Phi_p(V_p) \cap \Phi_q(V_q)$, where p and q are two paths of length l running through the vertices $(e(1), \dots, e(l+1))$ in that order. We prove that, if X and Y are irreducible components of V_p and V_q, respectively, such that $\Phi_p(X) \cap \Phi_q(Y) \neq \emptyset$, then

X and Y are birationally equivalent, and consequently one of them can be deleted from $V_{\mathbb{S}}$ as 'a double' without loss of information (Section 5).

In case the path $p \in K\Gamma$ does not start with an oriented cycle, the surjection

$$\Phi_p : V_p \to \{\text{isomorphism types of uniserials in } \Lambda\text{-mod with mast } p\}$$

is a bijection; as a consequence, the points of V_p serve as complete isomorphism invariants for the uniserial modules with mast p in this situation. On the other hand, Φ_p may fail to be bijective in the presence of certain types of oriented cycles within p. In Section 4, we give an algebraic characterization of the fibres of the maps Φ_p, thus providing a general solution to the isomorphism problem. It turns out that there is a system of equations with the following property: on insertion of arbitrary points P and Q of V_p, it specializes to a *linear* system in the remaining variables, the consistency of which is equivalent to '$\Phi_p(P) \cong \Phi_p(Q)$'. This classification is quite gratifying since the varieties V_p, as well as the pertinent systems of equations, are very accessible – they can be readily computed on the basis of quiver and relations for Λ – and the points P of V_p store information on the modules $\Phi_p(P)$ in an easily decodable form. We note that Bongartz proved in [4] that the isomorphism problem for any pair of finite dimensional modules can be resolved in a finite number of steps through a Gaussian elimination process. In contrast, our procedure, being tailored specifically for uniserial modules, avoids the complications involved in handling arbitrary finite dimensional modules.

In a nutshell, Section 6 is devoted to showing that *every* affine variety over K arises as the geometric quotient, modulo the GL-action, of the open subvariety of $\mathbb{M}\mathrm{od}_{l+1}(\Lambda)$ consisting of the points representing uniserial modules with fixed sequence $\mathbb{S}$ of composition factors. In our present terminology: Given *any* affine variety V over K, the family of irreducible components of V can be realized as a $V_{\mathbb{S}}$ for a suitable finite dimensional algebra $\Lambda = K\Gamma/I$ – with Γ acyclic – and a suitable sequence $\mathbb{S}$ of simple Λ-modules. We can even assume that Γ is without double arrows, which implies that the family $V_{\mathbb{S}}$ consists precisely of the irreducible components of a single variety V_p which is isomorphic to V on one hand, and uniquely determined up to isomorphism by Λ on the other. Recall, moreover, that acyclicity of Γ entails bijectivity of the corresponding natural map Φ_p as discussed above.

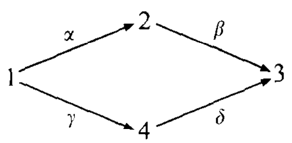
2. Preliminaries

Throughout, K will stand for an arbitrary field, and $\Lambda \cong K\Gamma/I$ will be a finite dimensional path algebra modulo relations over K; here Γ is a quiver and I is an admissible ideal in the path algebra $K\Gamma$. Our convention for the composition of paths $p, q \in K\Gamma$ is as in [2], namely, qp stands for 'q after p' whenever the concatenation is defined. Moreover, J will denote the Jacobson radical of Λ. As is well known, the ideal I factored out of $K\Gamma$ is, in general, not an isomorphism invariant of Λ, but depends on the choice of a complete set $e_1, \ldots, e_n$ of primitive idempotents and on the choice of 'arrows' from e_i to e_j, that is, of elements from e_jJe_i which give rise to a K-basis for $e_jJe_i/e_jJ^2e_i$. Any choice of an ideal I in $K\Gamma$, together with an isomorphism from $K\Gamma/I$ onto Λ, will be called a *coordinatization* of Λ. For simplicity, we will assume that Λ is equal to $K\Gamma/I$, unless otherwise specified, and identify the vertices of Γ with the corresponding primitive idempotents $e_1,\dots,e_n$ of Λ. Our Λ-modules will be *left* modules throughout.

Definition 1. Given a uniserial Λ-module U of length $l+1$, any path p of length l in $K\Gamma$ with $pU \neq 0$ is called a *mast* of U.

Elementary observations. (1) If the quiver Γ has no double arrows, then each uniserial Λ-module has a unique mast. Conversely, the uniqueness of masts in all uniserial modules implies absence of double arrows.

(2) Not every path in $K\Gamma$ with nonzero image in Λ needs to occur as a mast of a uniserial module. For example, if $\Lambda = K\Gamma/I$, where Γ is the quiver

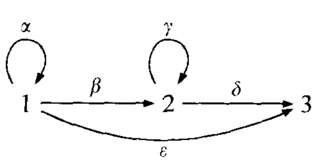


and $I = \langle \beta\alpha - \delta\gamma \rangle$, then neither $\beta\alpha$ nor $\delta\gamma$ is a mast of a uniserial Λ-module.

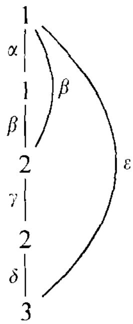
(3) Of course, this concept of mast depends on a given coordinatization of Λ. Our choice of coordinates will therefore impinge on the varieties of uniserials with given mast, to be introduced in the next section. As a consequence, the effect of a coordinate change needs to be discussed (see [5]).

Let M be a Λ-module. A *top element* of M is an element $x \in M\backslash JM$ with $e_ix = x$ for some $i \in \{1,\dots,n\}$; in that case, x will also be called a *top element of type* e_i. Clearly, given a uniserial Λ-module U of length $l+1$ with top element x, a path $p \in K\Gamma$ of length l is a mast of U if and only if $px \neq 0$. A useful tool in visualizing and communicating uniserial modules will be their *labeled and layered graphs*. Suppose that $p = \alpha_l \cdots \alpha_1$ where each arrow α_i has starting point $e(i)$ and endpoint $e(i+1)$. The labeled and layered graph of U with respect to a top element x and a mast p consists of (a) the mast p, drawn vertically with $e(1)$ at the top and edges labeled by the arrows α_i, together with (b) an edge labeled ω from $e(i)$ to $e(j)$ whenever $i < j$ and ω is an arrow from $e(i)$ to $e(j)$ such that $\Lambda\omega\alpha_{i-1}\cdots\alpha_1 x = J^{j-1}U$.

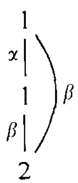
For example, let Γ be the quiver

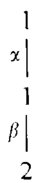


and $\Lambda = K\Gamma/\langle \alpha^2, \gamma^2 \rangle$. That the (layered and labeled) graph of a uniserial module U with top element x be



means that U has mast $p = \delta\gamma\beta\alpha$, that $\Lambda\beta x = J^2 U$, and $\Lambda\varepsilon x = J^4 U$; in other words, βx is congruent to a nonzero scalar multiple of $\beta\alpha x$ modulo $J^3 U$, and εx is a nonzero scalar multiple of px. Observe that these layered and labeled graphs are, in general, not completely determined by the isomorphism types of the modules they represent, but may depend on the choice of top element. For instance, if Λ is as above and $U = \Lambda e_1/(\Lambda\gamma\beta\alpha + \Lambda(\beta - \beta\alpha) + \Lambda\varepsilon)$, then the graphs of U relative to the top elements $x = \bar{e}_1$ and $y = \bar{e}_1 - \alpha\bar{e}_1$ are

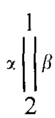


and


respectively. Conversely, layered and labeled graphs of uniserial modules do not pin these modules down up to isomorphism. Indeed, if $\Lambda = K\Gamma$, where Γ is the Kronecker quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$$

then any uniserial module $U_k = \Lambda e_1/\Lambda(\beta - k\alpha)$ for $k \in K \setminus \{0\}$ has graph



while obviously $U_k \not\cong U_l$ for $k \neq l$.

Finally, we will speak of *subpaths* of a path $p \in K\Gamma$: A path $q \in K\Gamma$ is a *right subpath* (respectively, *left subpath*) of p if there exists a path r with $p = rq$ (respectively,

$p = qr$). In particular, if p is a path from a vertex $e(1)$ to a vertex $e(2)$, then $e(1)$ is the unique right subpath of length zero of p, and $e(2)$ is the unique left subpath of length zero. A *proper* right subpath of p is a right subpath which is strictly shorter than p. It will be convenient to communicate the statement 'q is a right subpath of p' in the form '$p = \bullet q$' to arrive at compact formulas.

For the geometric terms we use, the reader should consult the introductory texts by Hartshorne and Mumford [7, 10].

3. Description of the varieties of uniserials with fixed mast

Throughout this section, let p be a path of length l from a vertex $e(1)$ to a vertex $e(l+1)$. We will use p to denote both the element in $K\Gamma$ and its residue class in Λ, unless there is a danger of ambiguity.

Roughly speaking, the affine K-variety V_p corresponding to the uniserial Λ-modules with mast p consists of points that are families of coordinate vectors of the following type: Given a uniserial left Λ-module U with mast p and top element x, we equip U with the K-basis $p_i x$, for $p = \bullet p_i$, and string up the coordinate vectors of the elements qx, where q runs through the paths in $K\Gamma$ not vanishing in Λ. Of course, these coordinate strings pin down the corresponding uniserials. The first point to be addressed is the fact that these strings of coordinate vectors actually form an affine algebraic variety V_p which, in fact, is readily accessible on the basis of a coordinatization of Λ. A crude outline of the procedure for assembling a finite set of polynomials which defines V_p is as follows: Replace the scalars $k_{i,q}$ arising in the linear dependence relations $qx = \sum_{p=\bullet p_i} k_{i,q} p_i x$ inside an arbitrary uniserial module with mast p and top element x by independent indeterminates $X_{i,q}$, and expand a representative set of relations from I by means of substitutions $q = \sum_{p=\bullet p_i} X_{i,q} p_i$ inside the polynomial ring $K\Gamma[X_{i,q}]$. A repetition of this substitution process will eventually reduce the relations to equations of the form $\sum_{p=\bullet p_i} \tau_i p_i = 0$, where the τ_i are polynomials in $K[X_{i,q}]$. Reflecting the fact that the elements $p_i x$ are K-linearly independent, our interest will be in the simultaneous vanishing set of these polynomials τ_i. What makes the resulting affine variety fairly manageable is the fact that we can significantly reduce the set of variables without renouncing information.

Next we give a formal description of the variety V_p and the canonical map Φ_p from V_p onto the set of isomorphism classes of uniserial Λ-modules with mast p. Instead of considering all the indeterminates $X_{i,q}$ mentioned above, we will restrict our attention to those of the form $X_{i,\alpha u}$, where (α, u) is a 'detour' as defined below.

Definition 2. (1) A *detour on the path* p is a pair (α, u), where α is an arrow and u is a right subpath of p (length 0 being allowed) such that

(i) $\alpha u \neq 0$ in $K\Gamma$,

(ii) αu is not a right subpath of p in $K\Gamma$, but

(iii) there exists a right subpath v of p with $\text{length}(v) \geq \text{length}(u) + 1$ such that the endpoint of v coincides with the endpoint of α.

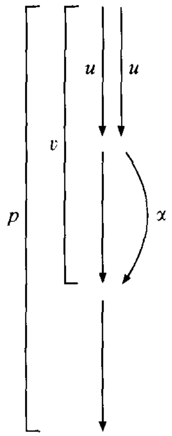


Fig. 1.

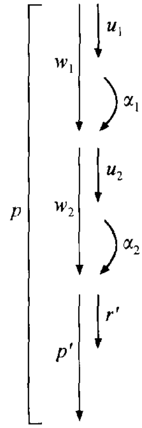


Fig. 2.

See Fig. 1. For ease of notation, we will often abbreviate the statement '(α, u) is a detour on p' by '$(\alpha, u) \wr\wr p$'.

(2) Suppose that p is a path of length l that passes consecutively through the vertices $e(1), \dots, e(l+1)$. A *route on* p is any path in $K\Gamma$ which starts in $e(1)$ and passes through a subsequence of the sequence $(e(1), \dots, e(l+1))$. (We include $e(1)$ and p in the set of routes on p.)

Remarks. (a) In particular, each right subpath of p is a route on p. More generally, any right subpath of a route is again a route.

(b) In our work with routes, the following factorization property will be crucial: A path $\mathbf{r} \in K\Gamma$ is a route on p if and only if it can be written in the form

$$\mathbf{r} = \mathbf{r}' \alpha_m u_m \cdots \alpha_1 u_1$$

for some $m \geq 0$ such that there exists a corresponding factorization

$$p = p' \cdot w_m \cdots w_1$$

of p with the property that (α_i, u_i) is a detour on w_i with endpoint(α_i) = endpoint(w_i) for each $i \leq m$, and $\mathbf{r}'$ is a right subpath of p'. (See Fig. 2.) Note that such a factorization of p corresponding to the given route $\mathbf{r}$ need not be unique.

(c) The length of any route $\mathbf{r}$ on p is bounded above by length(p).

(d) Whenever $\mathbf{r}w$ is a route on pw, then $\mathbf{r}$ is a route on p.

We start by describing the polynomial ring in which we will be working. Given any detour (α, u) on p, let

$$V(\alpha, u) = \{v_i(\alpha, u) \mid i \in I(\alpha, u)\}$$

be the family of right subpaths of p in $K\Gamma$ which are longer than u and have the same endpoint as α. In most cases, it will be more convenient to refer to the index set $I(\alpha,u)$ for $V(\alpha,u)$ than to the set $V(\alpha,u)$ itself. The polynomial ring of our choice will then be

$$K\Gamma[X] = K\Gamma[X_i(\alpha,u) \mid i \in I(\alpha,u),\ (\alpha,u) \wr\wr p]$$

with coefficients in the path algebra $K\Gamma$ and independent variables $X_i(\alpha,u)$. Next we introduce an equivalence relation on $K\Gamma[X]$ as follows: Let $\mathscr{L}(p)$ be the left ideal of $K\Gamma[X]$ generated by all the paths q in $K\Gamma$ which fail to be routes on p together with all the differences

$$\alpha u - \sum_{i \in I(\alpha,u)} X_i(\alpha,u) \cdot v_i(\alpha,u)$$

for detours (α,u) on p. Then, clearly, the relation '$\sigma \mathrel{\widehat{=}} \tau \Leftrightarrow \sigma - \tau \in \mathscr{L}(p)$' for $\sigma,\tau \in K\Gamma[X]$ defines a congruence relation relative to addition and left multiplication. The proof of the following easy observation is left to the reader.

Observation 3. *Each element of the path algebra $K\Gamma$ is $\widehat{=}$-congruent to a unique element of the form $\sum_{p=\bullet p'} \tau_{p'}(X)p'$, where the $\tau_{p'}(X)$ are polynomials in*

$$K[X] = K[X_i(\alpha,u) \mid i \in I(\alpha,u),\ (\alpha,u) \wr\wr p].$$

To obtain these polynomials $\tau_{p'}(X)$ for a given element $z \in K\Gamma$ algorithmically, consider the following *substitution equations for* p: First, $q \mathrel{\widehat{=}} 0$ for any path q in $K\Gamma$ which fails to be a route on p, and second,

$$\alpha u \mathrel{\widehat{=}} \sum_{i \in I(\alpha,u)} X_i(\alpha,u) \cdot v_i(\alpha,u)$$

for all detours (α,u) on p. We use the phrase 'inserting the substitution equations from the right' for the following steps:

$$q''q' \mathrel{\widehat{=}} \begin{cases} 0 & \text{if } q' \text{ is a nonroute on } p, \\ \sum_{i \in I(\alpha,u)} X_i(\alpha,u) \cdot q''v_i(\alpha,u) & \text{if } q' = \alpha u \text{ with } (\alpha,u) \wr\wr p. \end{cases}$$

Note that if q' is not a route on p, then neither is $q''q'$. Inserting the substitution equations from the right into the paths occurring in an element $z \in K\Gamma$ and repeating this procedure clearly leads to the equivalence $z \mathrel{\widehat{=}} \sum \tau_{p'}(X)p'$ with $\tau_{p'}(X) \in K[X]$ after at most d steps, where d is an upper bound on the lengths of the paths involved in z.

Now let L be the Loewy length of Λ – i.e., L is minimal with respect to $J^L = 0$ – and denote by $I^{(L)}$ the K-subspace of I consisting of all elements which can be written as K-linear combinations of paths of lengths at most L. Moreover, choose a finite K-generating set $t_1,\dots,t_s$ for the space $I^{(L)}$. By the above remarks, there are unique polynomials $\tau_{i,p'}(X) \in K[X]$ with the property that $t_i \mathrel{\widehat{=}} \sum_{p=\bullet p'} \tau_{i,p'}(X)p'$ for $1 \le i \le s$. We now give a definition of the variety V_p depending on this choice of relations

$t_1, \ldots, t_s$; this is the most convenient description for purposes of computation. However, as we will immediately observe, this definition of V_p is independent of the choice of the t_i.

Definition 4. Let $V_p = V(\tau_{i,p'}(X) \mid 1 \le i \le s$ and $p = \bullet p')$ be the simultaneous vanishing locus of the polynomials $\tau_{i,p'}(X)$ in affine N-space $\mathbb{A}^N = \mathbb{A}^N(K)$, where $N = \sum_{(\alpha,u) \wr p} |I(\alpha,u)|$.

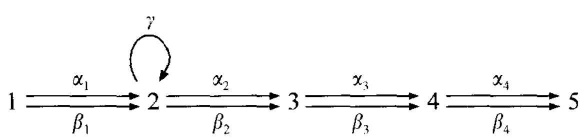
Remarks. (1) The affine algebraic set V_p is independent of the choice of a K-generating set for the K-space $I^{(L)}$; in fact, V_p is the vanishing locus of *all* polynomials $\tau_{p'}(X)$ arising in congruences $z \widehat{=} \sum_{p=\bullet p'} \tau_{p'}(X)p'$ for elements z from the ideal I. The former assertion is an immediate consequence of our construction, while the latter can easily be checked as follows: If the length of p exceeds the Loewy length L of Λ, then V_p is the empty set, because the right subpath p_L of p of length L belongs to $I^{(L)}$, and hence the congruence $p_L \widehat{=} 1 \cdot p_L$ places the constant 1 into the ideal of V_p. If, on the other hand, p has length $\le L$, all paths in $K\Gamma$ of length greater than L are non-routes on p and are hence reduced to zero under our equivalence relation.

(2) Our next remark along this line often saves a considerable amount of computational effort. Namely, observe that, whenever $r_1, \ldots, r_m \in I$ generate I as a *left* ideal of $K\Gamma$, then $V_p = V(\rho_{i,q}(X) \mid 1 \le i \le m,\ p = \bullet q)$, where $r_i \widehat{=} \sum_{p=\bullet q} \rho_{i,q}(X)q$ with $\rho_{i,q}(X) \in K[X]$.

(3) A priori, the family of algebraic varieties V_p, where p runs through the set of paths in $K\Gamma$ which do not vanish in Λ, clearly depends on the chosen coordinatization of the split algebra Λ, that is, on a fixed set $e_1, \ldots, e_n$ of primitive idempotents and K-bases for the spaces $e_iJe_j/e_iJ^2e_j$. In fact, both the labels of the varieties considered and their realization in affine space depend on the choice of coordinates. This leads us to the following uniqueness problem: If $K\Gamma/I \cong K\Gamma/I'$, are the varieties V_p and V_p' formed relative to the two ideals I and I' isomorphic? While in general there is uniqueness only up to birational equivalence (see Section 5), the answer to this isomorphism question is positive for large classes of algebras (see [5]).

It is easy to compute the varieties V_p from a given coordinatization, i.e., from a presentation of Λ in terms of quiver and relations. These varieties will permit us to classify the uniserial modules in terms of the correspondence described in the main theorem of this section (Theorem A). Before we state this theorem, we illustrate the construction of the varieties V_p.

Example 5. Let $\Lambda = K\Gamma/I$, where Γ is the quiver



and I is the ideal of $K\Gamma$ generated by the following relations:

$$\gamma^3,\quad \beta_2\alpha_1,\quad \beta_2\gamma\alpha_1,\quad \alpha_4\alpha_3\beta_2\beta_1,\quad \alpha_4\alpha_3\beta_2\gamma^2\alpha_1 - \alpha_4\alpha_3\alpha_2\gamma^2\alpha_1,$$

$$\beta_4\beta_3\alpha_2\gamma\alpha_1 - \alpha_4\alpha_3\alpha_2\beta_1,\quad \alpha_4\alpha_3\alpha_2\gamma^2\beta_1 - \alpha_4\alpha_3\alpha_2\alpha_1,\quad \alpha_4\alpha_3\alpha_2\gamma\alpha_1 + \alpha_4\alpha_3\beta_2\gamma\beta_1,$$

$$\alpha_4\alpha_3\alpha_2\gamma\alpha_1 - \alpha_4\beta_3\alpha_2\gamma^2\alpha_1,\quad \alpha_4\alpha_3\alpha_2\gamma\alpha_1 - \beta_4\alpha_3\alpha_2\gamma^2\alpha_1.$$

Moreover, consider the path $p = \alpha_4\alpha_3\alpha_2\gamma^2\alpha_1$. To compute the variety V_p, observe that the detours on p and the corresponding substitution equations are as follows:

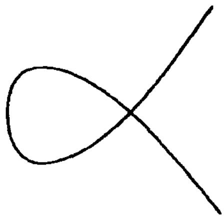
(β_1, e_1)	$\beta_1 \mathrel{\widehat{=}} X_1\alpha_1 + X_2\gamma\alpha_1 + X_3\gamma^2\alpha_1,$
(α_2, α_1)	$\alpha_2\alpha_1 \mathrel{\widehat{=}} X_4\alpha_2\gamma^2\alpha_1,$
(β_2, α_1)	$\beta_2\alpha_1 \mathrel{\widehat{=}} X_5\alpha_2\gamma^2\alpha_1,$
$(\alpha_2, \gamma\alpha_1)$	$\alpha_2\gamma\alpha_1 \mathrel{\widehat{=}} X_6\alpha_2\gamma^2\alpha_1,$
$(\beta_2, \gamma\alpha_1)$	$\beta_2\gamma\alpha_1 \mathrel{\widehat{=}} X_7\alpha_2\gamma^2\alpha_1,$
$(\beta_2, \gamma^2\alpha_1)$	$\beta_2\gamma^2\alpha_1 \mathrel{\widehat{=}} X_8\alpha_2\gamma^2\alpha_1,$
$(\beta_3, \alpha_2\gamma^2\alpha_1)$	$\beta_3\alpha_2\gamma^2\alpha_1 \mathrel{\widehat{=}} X_9\alpha_3\alpha_2\gamma^2\alpha_1,$
$(\beta_4, \alpha_3\alpha_2\gamma^2\alpha_1)$	$\beta_4\alpha_3\alpha_2\gamma^2\alpha_1 \mathrel{\widehat{=}} X_{10}p.$

Next observe that the relations listed above, together with the paths $\gamma^3\alpha_1$ and $\gamma^3\beta_1$, generate I as a left ideal of $K\Gamma$; consequently, Remark 2 following Definition 4 tells us that we need only consider these elements of I in determining a generating set of polynomials for the ideal of V_p. Since these last two paths, as well as γ^3, are nonroutes on p, they are $\widehat{=}$-equivalent to 0 and hence do not lead to conditions on the indeterminates X_i. We now insert the substitution equations into the remaining relations.

First $\beta_2\alpha_1 \mathrel{\widehat{=}} X_5\alpha_2\gamma^2\alpha_1$, which, in view of $\beta_2\alpha_1 \in I$, gives us the equation $X_5 = 0$ for V_p. Analogously, the combination of $\beta_2\gamma\alpha_1 \mathrel{\widehat{=}} X_7\alpha_2\gamma^2\alpha_1$ and $\beta_2\gamma\alpha_1 \in I$ implies $X_7 = 0$ for the points of V_p. Moreover, $\alpha_4\alpha_3\beta_2\beta_1 \mathrel{\widehat{=}} \alpha_4\alpha_3\beta_2(X_1\alpha_1 + X_2\gamma\alpha_1 + X_3\gamma^2\alpha_1) \mathrel{\widehat{=}} X_1X_5p + X_2X_7p + X_3X_8p$ yields $X_1X_5 + X_2X_7 + X_3X_8 = 0$ on V_p. In view of $X_5 = X_7 = 0$, we obtain $X_3X_8 = 0$. We further compute $\alpha_4\alpha_3\beta_2\gamma^2\alpha_1 - p \mathrel{\widehat{=}} (X_8 - 1)p$ to conclude that $X_8 - 1 = 0$, and consequently also $X_3 = 0$, on V_p. Next, $\beta_4\beta_3\alpha_2\gamma\alpha_1 - \alpha_4\alpha_3\alpha_2\beta_1 \mathrel{\widehat{=}} X_6\beta_4\beta_3\alpha_2\gamma^2\alpha_1 - X_1\alpha_4\alpha_3\alpha_2\alpha_1 - X_2\alpha_4\alpha_3\alpha_2\gamma\alpha_1 - X_3\alpha_4\alpha_3\alpha_2\gamma^2\alpha_1 \mathrel{\widehat{=}} X_6X_9X_{10}p - X_1X_4p - X_2X_6p - X_3p$, which gives us $X_6X_9X_{10} - X_1X_4 - X_2X_6 - X_3 = 0$, and $X_6X_9X_{10} - X_1X_4 - X_2X_6 = 0$ in view of the preceding equations. Analogously, the last four relations yield $X_1 - X_4 = 0$, $X_6 + X_1X_7 + X_2X_8 = X_6 + X_2 = 0$, $X_6 - X_9 = 0$, and $X_6 - X_{10} = 0$, respectively. We deduce that

$$V_p = V(X_3, X_5, X_7, X_8 - 1, X_6X_9X_{10} - X_1X_4 - X_2X_6,$$
$$X_1 - X_4, X_2 + X_6, X_6 - X_9, X_6 - X_{10}).$$

Clearly, V_p is isomorphic to the variety $V(Y^2 - X^3 - X^2)$ in affine 2-space. Over the real numbers, we thus obtain the standard α-curve:



In this example, the correspondence between the points of V_p and the isomorphism types of uniserial Λ-modules with mast p, as described in Theorem A below, is bijective, even though our path p does include an oriented cycle.

The following theorem contains basic information about the 'back and forth' between points on the variety V_p on one hand and uniserial Λ-modules with mast p on the other.

Theorem A. *Suppose that p is a path in $K\Gamma$ starting in the vertex $e(1)$.*

(I) *There is a surjective map Φ_p from the variety V_p to the set of isomorphism types of uniserial left Λ-modules with mast p. It assigns to each point $k = (k_i(\alpha,u))_{i\in I(\alpha,u),(\alpha,u)\wr p}$ in V_p the isomorphism type of the module $\Lambda e(1)/U_k$, where*

$$U_k = \left(\sum_{(\alpha,u)\wr p} \Lambda\left(\alpha u - \sum_{i\in I(\alpha,u)} k_i(\alpha,u)v_i(\alpha,u)\right)\right) + \left(\sum_{q \text{ not a route on } p} \Lambda q e(1)\right).$$

Alternately, the uniserial module $\Lambda e(1)/U_k$ representing $\Phi_p(k)$ can be described as the unique uniserial factor module with mast p of the module

$$\Lambda e(1) \Big/ \left(\sum_{(\alpha,u)\wr p} \Lambda\left(\alpha u - \sum_{i\in I(\alpha,u)} k_i(\alpha,u)v_i(\alpha,u)\right)\right).$$

(II) *The variety V_p is nonempty if and only if there exists a uniserial left Λ-module with mast p.*

(III) *Provided that p does not have a proper right subpath which is an oriented cycle of positive length, the map Φ_p is bijective.*

Proof. The following maps will repeatedly prove useful. For any family of scalars $k = (k_i(\alpha,u))_{i\in I(\alpha,u),(\alpha,u)\wr p}$ in $\mathbb{A}^N$, we will consider a K-linear transformation $F_k : K\Gamma \to K\Gamma$ depending on k: Namely, F_k sends all the paths which fail to be routes on p to zero and acts on routes q as follows: If q is a right subpath of p, then $F_k(q) = q$, and if $q = q'\alpha u$, where (α,u) is a detour on p, then $F_k(q) = \sum_{i\in I(\alpha,u)} k_i(\alpha,u)q'v_i(\alpha,u)$. Then, given any element $z \in K\Gamma$ which is a K-linear combination of paths of length at most m, the image of z under $(F_k)^m$ equals $\sum_{p=\bullet p'} \tau_{p'}(k)p'$, where $\sum_{p=\bullet p'} \tau_{p'}(X)p'$ is the unique element in $\sum_{p=\bullet p'} K[X]p'$ equivalent to z under '$\widehat{=}$'; here we are working inside the polynomial ring $K\Gamma[X]$, where X stands for the family of variables $X_i(\alpha,u)$. In particular, if z belongs to I, the polynomials $\tau_{p'}(X)$ fall into the ideal of V_p (by the first of the above remarks) and consequently vanish at k whenever $k \in V_p$; in other words, $(F_k)^m(z) = 0$ in that case.

To any family $k = (k_i(\alpha,u)) \in \mathbb{A}^N$ as above, we moreover assign the left ideal W_k of the path algebra $K\Gamma$, generated by the paths q which are nonroutes on p and by the elements $\alpha u - \sum_{i\in I(\alpha,u)} k_i(\alpha,u)v_i(\alpha,u)$, where (α,u) runs through the detours on p. Observe that, for any element $z \in K\Gamma$ and any choice of k, the difference $z - F_k(z)$ belongs to W_k; in particular, the subspace W_k of $K\Gamma$ is invariant under F_k.

Let $\Phi = \Phi_p$ be defined as in the statement of the theorem and, for $k \in V_p$, identify $\Phi(k)$ with the module $\Lambda e(1)/U_k$.

Claim 1. *For each $k \in V_p$, the module $\Phi(k)$ is uniserial with mast p.*

Indeed, write $p = \beta_l \cdots \beta_1$, where the β_i are arrows, and set $x = e(1) + U_k \in \Phi(k)$. It clearly suffices to show that $\Phi(k)$ has K-basis $\beta_i \cdots \beta_1 x$, $0 \le i \le l$. Indeed, once this is established, we have $J^l\Phi(k) \neq 0$, and, for reasons of dimension, the radical powers $J^i\Phi(k)$, $0 \le i \le l$, constitute a composition series for $\Phi(k)$.

To see that the elements $\beta_i \cdots \beta_1 x$, $0 \le i \le l$, form a K-generating set for $\Phi(k)$, it suffices to check that, given any route $\mathbf{r}$ on p, the element $\mathbf{r}x \in \Phi(k)$ is a K-linear combination of the $\beta_i \cdots \beta_1 x$. We prove this by induction on $d = l - \text{length}(u_1)$, where u_1 is the right subpath of $\mathbf{r}$ of maximal length occurring also as a right subpath of p. If $d = 0$, then $\mathbf{r} = p$, and there is nothing to prove. So suppose $d \ge 1$, and let $\mathbf{r} = \mathbf{r}'\alpha_s u_s \cdots \alpha_1 u_1$. If $s = 0$, then $\mathbf{r}$ is a (proper) right subpath of p, and we are again done. If, on the other hand, $s \ge 1$, the definition of $\Phi(k)$ yields

$$\mathbf{r}x = \sum_{i\in I(\alpha_1,u_1)} k_i(\alpha_1,u_1)\mathbf{r}'\alpha_s u_s \cdots \alpha_2 u_2 v_i(\alpha_1,u_1)x.$$

If the path $\mathbf{r}'\alpha_s u_s \cdots \alpha_2 u_2 v_i(\alpha_1,u_1)$ is not a route on p, then $\mathbf{r}'\alpha_s u_s \cdots \alpha_2 u_2 v_i(\alpha_1,u_1)x = 0$ by construction. The induction hypothesis applies to each of the remaining terms $\mathbf{r}'\alpha_s u_s \cdots \alpha_2 u_2 v_i(\alpha_1,u_1)x$, since the lengths of the right subpaths $v_i(\alpha_1,u_1)$ of p exceed the length of u_1.

To prove that the elements $\beta_i \cdots \beta_1 x$ for $0 \le i \le l$ are linearly independent, it suffices to verify that $px = \beta_l \cdots \beta_1 x \neq 0$. Assume the contrary, and consider the following presentation of $\Phi(k)$ as a $K\Gamma$-module: $\Phi(k) = K\Gamma e(1)/(Ie(1) + W_k e(1))$, where W_k is as defined above. So $px = 0$ is equivalent to the existence of an element $a \in Ie(1)$ such that $p + a \in W_k$. To reach a contradiction, we will first derive that $p \in W_k$. Suppose that a is a K-linear combination of paths of lengths at most m. In view of the fact that $a \in I$, the first paragraph of the proof shows that $F_k^m(a) = 0$. Moreover, due to the F_k-invariance of W_k, the fact that $p + a \in W_k$ yields $p = F_k^m(p + a) \in W_k$ as required. But the containment $p \in W_k$ is in turn impossible, as we now prove.

Assume that p is a K-linear combination of terms $q'(\alpha u - \sum_{i\in I(\alpha,u)} k_i(\alpha,u)v_i(\alpha,u))$ and terms $q''q$, where q is not a route on p and q', q'' are paths in $K\Gamma$. Since none of the paths of the form αu going with a detour (α,u) on p is a right subpath of p, the above equality forces p to be equal to one of the terms $q'v_i(\alpha,u)$, say $p = q_0'v_i(\alpha_0,u_0)$. This clearly implies that $q_0'\alpha_0u_0$ is a route on p. Observe that, more generally, $q'\alpha u$ is a route on p, whenever (α,u) is a detour with the property that $q'v_j(\alpha,u)$ is a route on p for some $j\in I(\alpha,u)$. Let u_1 be a right subpath of p that has minimal length with respect to the following properties:

(a) for some arrow α_1, the pair (α_1,u_1) is a detour on p, and, for a suitable path q_1', the element $q_1'(\alpha_1u_1 - \sum k_i(\alpha_1,u_1)v_i(\alpha_1,u_1))$ occurs nontrivially in the above representation of p as an element of W_k;

(b) $q_1'\alpha_1u_1$ is a route on p.

Since α_1u_1 is not a right subpath of p and, a fortiori, $q_1'\alpha_1u_1 \neq p$, the term $q_1'\alpha_1u_1$ must cancel out of our representation of p as an element of W_k. Clearly $q_1'\alpha_1u_1 \neq q''q$ whenever q fails to be a route on p, and $q_1'\alpha_1u_1 \neq q_2'\alpha_2u_2$ whenever (α_2,u_2) is a detour different from (α_1,u_1). Thus $q_1'\alpha_1u_1 = q_2'v_j(\alpha_2,u_2)$ for some j and some detour (α_2,u_2) of p such that $q_2'(\alpha_2u_2 - \sum_{i\in I(\alpha_2,u_2)} k_i(\alpha_2,u_2)v_i(\alpha_2,u_2))$ occurs nontrivially in our representation of p. On one hand, this forces $q_2'v_j(\alpha_2,u_2)$, and consequently also $q_2'\alpha_2u_2$, to be a route on p, whence length$(u_2) \geq$ length(u_1) by our choice of u_1. On the other hand, the equality '$q_1'\alpha_1u_1 = q_2'v_j(\alpha_2,u_2)$' is only possible when length $v_j(\alpha_2,u_2) \leq$ length(u_1), for u_1 is the longest right subpath of p which is also a right subpath of $q_1'\alpha_1u_1$. But this, in turn, entails length$(u_2) <$ length(u_1). We have reached a contradiction, which proves that $p \notin W_k$, and which thus completes the proof of Claim 1.

Claim 2. *Whenever U in Λ-mod is a uniserial module with mast p, there exists a point $k\in V_p$ such that $U \cong \Phi(k)$.*

To find a suitable point $k\in V_p$, let $x = e(1)x$ be a top element of U; then the map $f: \Lambda e(1) \to U = \Lambda x$ which sends $e(1)$ to x is a projective cover of U. Again write $p = \beta_l \cdots \beta_1$, where the β_i are arrows. Then, clearly, $J^iU = \Lambda\beta_i\cdots\beta_1x$ for $i \leq l$, and the elements $\beta_i\cdots\beta_1x$, $0 \leq i \leq l$, form a K-basis for U.

If (α,u) is a detour on p, then $\Lambda\alpha ux = J^mx$ for some $m \geq$ length$(u)+1$. Consequently, letting e be the primitive idempotent with $e\alpha = \alpha$, we obtain that αux is a K-linear combination of those elements $\beta_i\cdots\beta_1x$ of U for which $\beta_i\cdots\beta_1$ is a path longer than u ending in e. But these latter paths are precisely the ones that belong to the family $(v_i(\alpha,u))_{i\in I(\alpha,u)}$, which shows that $\alpha ux = \sum_{i\in I(\alpha,u)} k_i(\alpha,u)v_i(\alpha,u)x$ for suitable scalars $k_i(\alpha,u)\in K$. (Note on the side: Once we have fixed a top element x of U, the scalars $k_i(\alpha,u)$ are actually uniquely determined by U, since the elements $v_i(\alpha,u)x$, $i\in I(\alpha,u)$, are K-linearly independent.) In other words, via the map f, the uniserial U is an epimorphic image of the module $M := \Lambda e(1)/\sum_{(\alpha,u)\wr p} \Lambda(\alpha u - \sum_{i\in I(\alpha,u)} k_i(\alpha,u)v_i(\alpha,u))$. Set $k = (k_i(\alpha,u))_{i\in I(\alpha,u),(\alpha,u)\wr p}$. In order to prove Claim 2, it is thus enough to show that (a) $qx = 0$, whenever $q\in K\Gamma$ is a path which fails to be a route on p, and (b) $k\in V_p$. For reasons of dimension, we will then obtain that the epimorphism $\Phi(k) \to U$

induced by f is an isomorphism. In particular, statement (a) will guarantee that, up to isomorphism, U is the only factor module of M which is uniserial with mast p.

For (a), suppose that the path q is a nonroute on p. It is clearly harmless to assume that $q = qe(1)$. Write $q = q'u$, where u is the longest right subpath of q (possibly of length zero) which *is* a route on p. Moreover, let p' be the shortest right subpath of p such that u is a route on p', say $p = p''p'$. Then length $q' \geq 1$, that is, $q' = q''\gamma$ for some arrow γ, and the fact that γu is not a route on p is equivalent to the nonexistence of a right subpath v of p which is strictly longer than p' and ends in the same vertex as γ. If $ux \neq 0$, then the minimal choice of p' implies that $\Lambda ux = \Lambda w'x$ where w' is a right subpath of p with length$(w') \geq$ length(p'). If $p = w''w'$, then w'' is a left subpath of p'' such that $\Lambda ux = \Lambda w'x$ is a uniserial module with mast w''. Consequently, the fact that p'', and hence also w'', is devoid of right subpaths of positive length ending in the same vertex as γ forces γux to be zero. Thus $qx = 0$ as required.

To verify that k is a point in V_p, let z be any element in I; say z is a linear combination of paths of lengths bounded above by some integer m. Moreover, let $\sum_{p=\bullet p'} \tau_{p'}(X)p'$ be the unique element in $\sum_{p=\bullet p'} K[X]p'$ which is equivalent to z under $\widehat{=}$. We want to show that all of the polynomials $\tau_{p'}(X)$ vanish at k. For that purpose, view U again as a $K\Gamma$-module under the action induced by that of Λ, and note that $zU = 0$, as well as $W_k x = 0$. Since $F_k(y) - y \in W_k$ for all $y \in K\Gamma$, we infer that $F_k^r(z)x = 0$ for all $r \geq 1$. But as we pointed out in the first paragraph, $F_k^m(z) = \sum_{p=\bullet p'} \tau_{p'}(k)p'$, and so, in particular, $\sum_{p=\bullet p'} \tau_{p'}(k)p'x = 0$. Now use the fact that the elements $p'x$, where p' runs through the right subpaths of p, are K-linearly independent, to conclude $\tau_{p'}(k) = 0$ for all p' as required. This completes the proof of part (I). Part (II) is an obvious consequence of part (I).

(III) In proving the surjectivity of Φ, we saw that, given a uniserial module U with mast p and a fixed top element x, there is a unique point $k = (k_i(\alpha, u)) \in \Phi^{-1}(U)$ such that $\alpha ux = \sum_{i \in I(\alpha,u)} k_i(\alpha, u)v_i(\alpha, u)x$ for each detour (α, u) on p. In other words, each subfamily $(k_i(\alpha, u))_{i \in I(\alpha,u)}$ of k is the coordinate vector of αux relative to the K-basis $v_i(\alpha, u)x$, $i \in I(\alpha, u)$, of $eJ^{\text{length}(u)+1}U$; here e is the primitive idempotent in which α terminates. Since this coordinate vector will not change if x is replaced by ax for some nonzero scalar a, we see that, if U has a unique top element, up to scalar factors – equivalently, if p is not of the form $p = p'c$ for a cycle c of positive length from $e(1)$ to $e(1)$ – the set $\Phi^{-1}(U)$ is a singleton, i.e., Φ is injective. This leaves us to deal with the case, where p is a cycle from $e(1)$ to $e(1)$ such that no proper right subpath of p is a cycle $e(1) \to e(1)$. In that case, an arbitrary top element y of U is of the form $y = ax + bpx$ for $a, b \in K$ and a nonzero, and since $\alpha uy = \alpha uax$ and $v_i(\alpha, u)y = v_i(\alpha, u)ax$ for each detour (α, u) on p, the coordinate vector of αuy with respect to the new basis is the same as that of αux with respect to the old basis. Again we deduce that Φ is injective. □

Note that, in case K is algebraically closed, Theorem A(II) provides us with an algorithmic procedure to decide whether a given path $p: e \to e'$ in $K\Gamma$ occurs as mast of a uniserial left Λ-module. Indeed, when combined with Hilbert's Nullstellensatz, Theorem A yields the following: There is a uniserial Λ-module with mast p if and only if, for some K-generating set $t_1, \dots, t_s$ for $I^{(L)}$, the resulting polynomials $\tau_{i,p'}(X)$, as in the definition of the variety V_p, generate a proper ideal of $K[X]$. The ensuing question, whether or not 1 belongs to the ideal generated by the $\tau_{i,p'}(X)$ is well-known to be decidable by way of the Groebner method.

We will extract an observation from the preceding argument which will be useful on several occasions. For a smooth formulation, we require the following notational convention: Given a uniserial module U with mast p and a detour (α, u) on p such that α ends in the vertex e, we denote by $U(\alpha, u)$ the K-subspace $eJ^{\text{length}(u)+1}U$ of U. Note that, for any top element x of U, the set $\{v_i(\alpha, u)x \mid i \in I(\alpha, u)\}$ forms a basis for this subspace.

Corollary to the proof of Theorem A. *Let N be the disjoint union of the index sets $I(\alpha,u)$, where (α,u) runs through the detours on p, and let k be a point in $\mathbb{A}^N(K)$, say $k = (k_i(\alpha,u))_{i \in I(\alpha,u),(\alpha,u) \wr p}$. Then k belongs to V_p precisely when there exists a uniserial module U with mast p and top element x such that, for each detour (α,u) on p, the projection $(k_i(\alpha,u))_{i\in I(\alpha,u)}$ of k onto $\mathbb{A}^{I(\alpha,u)}$ is the coordinate vector of the element $\alpha u x \in U(\alpha,u)$ with respect to the K-basis $\{v_i(\alpha,u)x \mid i \in I(\alpha,u)\}$.*

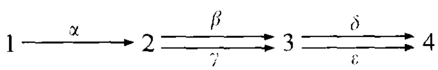
In the positive case, any such uniserial module U belongs to $\Phi_p(k)$, and the top element $x = e(1) + U_k$ of $\Lambda e(1)/U_k$ has the property that $\alpha u x = \sum_{i \in I(\alpha,u)} k_i(\alpha,u)v_i(\alpha,u)x$ for all $(\alpha,u) \wr p$; here U_k is as in the statement of Theorem A.

Let L again denote the Loewy length of Λ, and $I^{(L)}$ the K-subspace of I consisting of all elements of I which can be written as K-linear combinations of paths of lengths $\leq L$. In the definition of V_p, we picked a K-generating set for $I^{(L)}$ to arrive – via the substitution equations for p – at a set of polynomials that determines V_p. This set may be vastly redundant, as we already pointed out after Definition 4. In fact, it suffices to consider elements $t_1, \dots, t_r \in I^{(L)}$ such that $I^{(L)} \subseteq \sum_{i=1}^r K\Gamma t_i$; if again $\tau_{i,p'}(X) \in K[X]$ are such that $t_i \mathrel{\hat{=}} \sum_{p=\bullet p'} \tau_{i,p'}(X)p'$ for $1 \leq i \leq s$, then

$$V_p = V(\tau_{i,p'}(X) \mid 1 \leq i \leq r,\ 0 \leq j \leq l).$$

On the other hand, it does not suffice to consider a set of relations which generates I as an ideal, as the following example demonstrates.

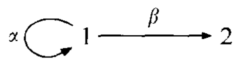
Example 6. Let $\Lambda = K\Gamma/I$, where Γ is the quiver



and $I \subseteq K\Gamma$ is the ideal generated by the relations $\delta\beta - \varepsilon\gamma$, $\varepsilon\beta$, and $\delta\gamma$. Moreover, let $p = \delta\beta\alpha$. The detours on p are (γ, α) and $(\varepsilon, \beta\alpha)$, and hence the substitution equations for p are $\gamma\alpha \mathrel{\hat{=}} X_1\beta\alpha$, $\varepsilon\beta\alpha \mathrel{\hat{=}} X_2 p$, as well as $q \mathrel{\hat{=}} 0$ whenever q fails to be a route on p. Note that none of the paths occurring in the above relations is a route on p, and so the polynomials resulting from these relations via the substitution equations are all zero. On the other hand, the relations $\delta\beta\alpha - \varepsilon\gamma\alpha$ and $\varepsilon\beta\alpha$ yield $1 - X_1X_2 = 0$ and $X_2 = 0$, whence $V_p = \emptyset$.

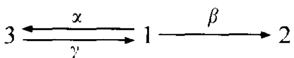
The following instance of failure of bijectivity of Φ_p is prototypical.

Example 7. Let $\Lambda = K\Gamma/\langle\alpha^2\rangle$, where Γ is the quiver



If $p = \beta\alpha$, then $V_p = \mathbb{A}^1$, while $\Phi_p(V_p)$ is a singleton; indeed, for any scalar $k \in K$, the uniserial modules $\Lambda e_1/\Lambda(\beta - k\beta\alpha)$ and $\Lambda e_1/\Lambda\beta$ are isomorphic.

However, the sufficient condition for bijectivity of Φ_p given in Theorem A(III) is not necessary. Indeed, if Γ is the quiver



and $\Lambda = K\Gamma/\langle\alpha\gamma\alpha\rangle$, the path $p = \beta\gamma\alpha$ again consists of a cycle followed by a nontrivial left subpath. Again we have $V_p = \mathbb{A}^1$, but this time the map Φ_p is bijective.

Definition 8. Again, let p be a path in $K\Gamma$ and $\Lambda = K\Gamma/I$. We will refer to V_p as the *uniserial variety of* Λ*-mod at* p. The irreducible components of the uniserial variety V_p will be called the *uniserial components of* Λ*-mod at* p. Moreover, we will say that a uniserial component W of V_p *intersects* V_q if $\Phi_p(W) \cap \mathrm{Im}(\Phi_q) \neq \emptyset$.

Let us start by looking at two trivial cases: If $p = e$ is a primitive idempotent, then V_p is a singleton, represented by the simple module centered at e, and if $p = \alpha$ is an arrow $e \to e'$, then $V_p = \mathbb{A}^n$, where n is the number of arrows $e \to e'$ different from α. In each of these cases, the uniserial variety at p consists of a single component.

Observe that, in case the quiver Γ of Λ has no double arrows between any two vertices, the images $\Phi_p(V_p)$ of the uniserial varieties, where p runs through the paths in $K\Gamma$, yield a disjoint partitioning of the set of isomorphism classes of uniserial objects in Λ-mod. In general, however, the uniserial varieties at p and q, where p and q are distinct paths of the same length passing through the same sequence of vertices, may intersect. We will see in Section 5 that the uniserial components at p which intersect V_q and those components at q which intersect V_p coincide in number and can be arranged into birationally equivalent pairs.

Two further problems impose themselves as follow-ups to the preceding theorem. One concerns functoriality of the assignment $\Lambda = K\Gamma/I \mapsto (V_p)_{p \text{ a path in } K\Gamma}$ and, in particular, the behavior of the family of varieties V_p under algebra isomorphism. The other is the isomorphism problem for uniserial modules. We defer the former to [5], and proceed by tackling the latter.

4. The isomorphism problem for uniserial modules

Since an obvious necessary condition for isomorphism of two uniserial modules is a joint mast, we wish to explicitly describe the equivalence relation on V_p which partitions V_p into the fibres $\Phi_p^{-1}(U)$, where U runs through the uniserial modules in the image $\Phi_p(V_p)$. Theorem A(III) provides us with a partial answer: If the path p does not start with an oriented cycle, i.e., if p is not of the form $p'c$ where c is an oriented cycle of positive length, the map Φ_p is a bijection; in other words, the points on the variety V_p form a complete system of isomorphism invariants for the uniserials with mast p in that case. In general, injectivity of Φ_p may fail, as we know from Example 7. To fill the resulting gap in the information on the uniserial modules with mast p stored in V_p, we will construct a system of equations $S_p(X,Y,Z)$ in the variables $X_i(\alpha,u)$, $Y_i(\alpha,u)$ (for $(\alpha,u) \wr\wr p$ and $i \in I(\alpha,u)$) and finitely many variables Z_j, which is linear in the Z_j over $K[X,Y]$ such that, for any pair of points $k,k' \in V_p$, the linear system $S_p(k,k',Z)$ is consistent if and only if $\Phi_p(k) \cong \Phi_p(k')$. So, loosely speaking, the family of uniserial modules with mast p can be identified with the variety V_p modulo a certain system of linear equations with coefficients in K. Since it is easy to establish, this system will provide a handy decision process for the isomorphism problem on the basis of the varieties V_p.

To describe the system $S_p(X,Y,Z)$, suppose that the path $p: e(1) \to e(l+1)$ has precisely t right subpaths of positive length ending in the starting vertex $e(1)$ of p, say $w_1,\dots,w_t$. Then our system will have the t linear variables $Z_1,\dots,Z_t$. Start by considering the following equations $E(\alpha,u)$ in $K\Gamma[X,Y,Z]$, one for each detour (α,u) on p:

$$(E(\alpha,u)) \qquad \alpha u\left(e(1)+\sum_{j=1}^{t} Z_j w_j\right) = \sum_{i\in I(\alpha,u)} X_i(\alpha,u)v_i(\alpha,u)\left(e(1)+\sum_{j=1}^{t} Z_j w_j\right);$$

here the $v_i(\alpha,u)$ are as in the definition of the substitution equations for p. Now expand both sides of these equations by successively inserting from the right the substitution equations $\beta v \mathrel{\hat{=}} \sum_{i\in I(\beta,v)} Y_i(\beta,v)v_i(\beta,v)$ for detours (β,v) on p, and the equivalences $q \mathrel{\hat{=}} 0$ for those paths $q \in K\Gamma$ which fail to be routes on p. As pointed out in Observation 3 of Section 3, the equation $E(\alpha,u)$ will eventually take on the form

$$\sum_{i\in I(\alpha,u)} a_i(X,Y,Z)v_i(\alpha,u) = \sum_{i\in I(\alpha,u)} b_i(X,Y,Z)v_i(\alpha,u)$$

for suitable polynomials $a_i(X,Y,Z)$, $b_i(X,Y,Z) \in K[X,Y,Z]$ which are uniquely determined by the left-hand and right-hand sides of equation $E(\alpha,u)$. Indeed, this is always the terminal stage of the substitution process, since each substitution step replaces a path q by a linear combination of paths of lengths $\geq$ length(q), all of which have the same endpoint as q. Now collect all of the equations of the form $a_i(X,Y,Z) = b_i(X,Y,Z)$, $i \in I(\alpha,u)$, arising in this way for arbitrary detours (α,u) on p, and label the resulting system $S_p(X,Y,Z)$. Observe that this system is polynomial in the X_j and Y_j, linear in the Z_j.

Theorem B. *For $k, k' \in V_p$, the linear system $S_p(k, k', Z)$ in $Z = (Z_1, \dots, Z_t)$ is consistent if and only if $\Phi_p(k) \cong \Phi_p(k')$.*

Proof. First suppose that there exists an isomorphism $f : \Phi_p(k) \to \Phi_p(k')$ of Λ-modules, and let x and y be top elements of $\Phi_p(k)$ and $\Phi_p(k')$, with the property that $\alpha u x = \sum_{i \in I(\alpha,u)} k_i(\alpha,u) v_i(\alpha,u) x$ and $\alpha u y = \sum_{i \in I(\alpha,u)} k_i'(\alpha,u) v_i(\alpha,u) y$, respectively, for any detour (α, u) on p (here we identify the isomorphism classes $\Phi_p(k)$ and $\Phi_p(k')$ with the distinguished representatives described in Theorem A). The equation $f(x) = e(1) f(x)$ clearly yields $f(x) = c_0 y + \sum_{j=1}^{t} c_j w_j y$ with $c_j \in K$ and $c_0 \neq 0$; without loss of generality, we may assume $c_0 = 1$. Let (α, u) be a detour on p such that α ends in the primitive idempotent e, and consider the following equality inside the K-vectorspace $e\Phi_p(k')$ which results from our isomorphism f:

$$(\dagger) \qquad \alpha u \left(y + \sum_{j=1}^{t} c_j w_j y \right) = \sum_{i \in I(\alpha,u)} k_i(\alpha,u) v_i(\alpha,u) \left(y + \sum_{j=1}^{t} c_j w_j y \right).$$

Clearly, the left-hand side of equality (†) reduces to the form

$$\sum_{i \in I(\alpha,u)} a_i(k, k', c) v_i(\alpha,u) y \in e\Phi_p(k'),$$

where $c = (c_1, \dots, c_t)$ and $a_i(X, Y, Z)$ is the polynomial occurring in the definition of the system $S_p(X, Y, Z)$; this is clear from the discussion prior to Theorem B. Analogously, the right-hand side of (†) is equal to the element

$$\sum_{i \in I(\alpha,u)} b_i(k, k', c) v_i(\alpha,u) y \in e\Phi_p(k')$$

with $b_i(X, Y, Z) \in K[X, Y, Z]$ as above, and consequently equality (†) reduces to

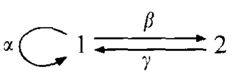
$$(\ddagger) \qquad \sum_{i \in I(\alpha,u)} a_i(k, k', c) v_i(\alpha,u) y = \sum_{i \in I(\alpha,u)} b_i(k, k', c) v_i(\alpha,u) y.$$

Since the vectors $v_i(\alpha,u) y$, $i \in I(\alpha,u)$, are K-linearly independent, this shows that the scalars $c_1, \dots, c_t$ satisfy the system $S_p(k, k', Z)$.

Conversely, suppose that the system $S_p(k, k', Z)$ is consistent, and let $c = (c_1, \dots, c_t) \in K^t$ be a solution. Moreover, denote the residue class of the vertex $e(1)$ in $\Phi_p(k)$ by x, that in $\Phi_p(k')$ by y (again we identify $\Phi_p(k)$ and $\Phi_p(k')$ with the representatives described in Theorem A). We wish to show that the assignment $x \mapsto y + \sum_{j=1}^{t} c_j w_j y$ extends to a well-defined Λ-isomorphism $\Phi_p(k) \to \Phi_p(k')$. It clearly suffices to show that the annihilator of x in Λ is contained in the annihilator of $y + \sum_{j=1}^{t} c_j w_j y$, since these two elements are top elements of the uniserials $\Phi_p(k)$ and $\Phi_p(k')$, respectively, and the uniserial modules $\Phi_p(k)$ and $\Phi_p(k')$ have the same length. Recall that, by the definiton of $\Phi_p(k)$, the left annihilator of x is, as a left ideal of Λ, generated by the (residue classes in Λ of) the elements $\alpha u - \sum_{i \in I(\alpha,u)} k_i(\alpha,u) v_i(\alpha,u)$, where (α, u) runs through the detours on p, and by all the (residue classes of) paths q which fail to be routes on p. That each of the former elements annihilates $y + \sum_{j=1}^{t} c_j w_j y$ is guaranteed by the fact that $c = (c_1, \dots, c_t)$ satisfies the system $S_p(k, k', Z)$. Indeed, from the fact that c satisfies (‡), we infer that c also satisfies equality (†) above. So let q be a nonroute on p. By construction, q also annihilates y; moreover, given any of the subpaths w_j of p ending in $e(1)$, the composition qw_j is not a route on p either by Remark (a) following Definition 2. This implies that also $qw_j y = 0$ for all j, and hence that $q(y + \sum_{j=1}^{t} c_j w_j y) = 0$ as required. □

We illustrate our method for solving the isomorphism problem with two examples. In the first, $V_p \cong \mathbb{A}^3$ and each of the fibres of Φ_p is a subvariety isomorphic to $\mathbb{A}^2$; in the second, $V_p \cong \mathbb{A}^1$ consists of a single fibre.

Example 9. Let $\Lambda = K\Gamma/I$, where Γ is the quiver



and I is the ideal in $K\Gamma$ generated by α^2, $\gamma\beta\gamma$, $\gamma\beta\alpha\gamma$. Consider the path $p = \beta\alpha\gamma\beta\alpha$. The detours on p are (β, e_1), $(\beta, \gamma\beta\alpha)$, and (α, α), yielding the substitution equations

$$\beta \mathrel{\widehat{=}} X_1\beta\alpha + X_2 p, \qquad \beta\gamma\beta\alpha \mathrel{\widehat{=}} X_3 p, \qquad \alpha^2 \mathrel{\widehat{=}} X_4\gamma\beta\alpha + X_5\alpha\gamma\beta\alpha.$$

Inserting the substitution equations from the right into a K-generating set for $I^{(6)}$ – note that 6 is the Loewy length of Λ – gives us $X_4 = X_5 = 0$, while imposing no conditions on X_1, X_2, X_3. Thus $V_p \cong \mathbb{A}^3$.

To determine the system $S_p(X, Y, Z)$, we observe that there are precisely three right subpaths of p of positive length which end in e_1, namely α, $\gamma\beta\alpha$, and $\alpha\gamma\beta\alpha$; thus $Z = (Z_1, Z_2, Z_3)$. Setting $z = e_1 + Z_1\alpha + Z_2\gamma\beta\alpha + Z_3\alpha\gamma\beta\alpha$ and inserting the substitution equations $\beta \mathrel{\widehat{=}} Y_1\beta\alpha + Y_2 p$, $\beta\gamma\beta\alpha \mathrel{\widehat{=}} Y_3 p$ and $\alpha^2 \mathrel{\widehat{=}} 0$ repeatedly into the three starting equations $E(\beta, e_1)$: $\beta z = X_1\beta\alpha z + X_2 pz$; $E(\beta, \gamma\beta\alpha)$: $\beta\gamma\beta\alpha z = X_3 pz$ and $E(\alpha, \alpha)$: $\alpha^2 z = X_4\gamma\beta\alpha z + X_5\alpha\gamma\beta\alpha z$ as described ahead of Theorem B, we obtain

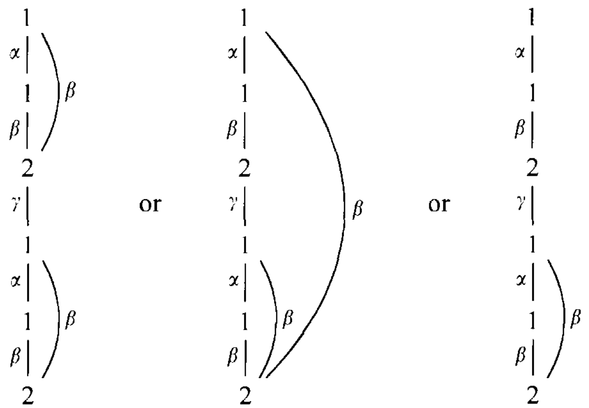
$$\begin{aligned} X_1\beta\alpha + Z_2X_1p + X_2p &= Y_1\beta\alpha + Y_2p + Z_1\beta\alpha + Z_2\beta\gamma\beta\alpha + Z_3p \\ &= Y_1\beta\alpha + Y_2p + Z_1\beta\alpha + Z_2Y_3p + Z_3p \end{aligned}$$

from equation $E(\beta, e_1)$, $X_3p = Y_3p$ from the second of the starting equations, and $0 = 0$ from the third. Thus the system $S_p(X, Y, Z)$ is

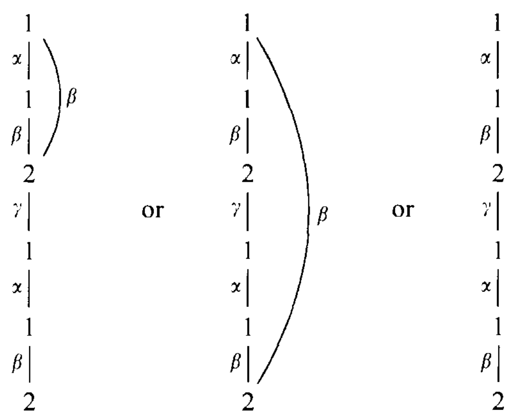
$$X_1 = Y_1 + Z_1, \qquad Z_2X_1 + X_2 = Y_2 + Z_2Y_3 + Z_3, \qquad X_3 = Y_3$$

in this example. In particular, given any two points $k = (k_1, k_2, k_3)$ and $k' = (k_1', k_2', k_3')$ in $V_p \cong \mathbb{A}^3$, the linear system $S_p(k, k', Z)$ in Z_1, Z_2, Z_3 is consistent if and only if $k_3 = k_3'$. This shows that, up to isomorphism, there is only a one-parameter family of uniserial modules with mast p, the parameter being $k_3 \in K$.

Let us translate this information into the graphs of the uniserial modules with mast p. Each of the uniserial modules corresponding to a nonzero value of k_3 has three different graphs, depending on the choice of top element. These graphs are:

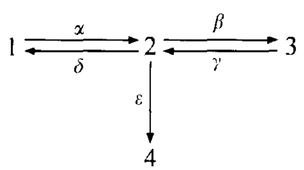


The uniserial module corresponding to the value $k_3 = 0$ has graphs



again depending on the choice of a top element.

Example 10. Let $\Lambda = K\Gamma/I$, where Γ is the quiver



and I is the ideal in $K\Gamma$ generated by the relations

$$\varepsilon\alpha - \varepsilon\gamma\beta\alpha,\ \delta\alpha\delta\alpha,\ \alpha\delta\alpha\delta,\ \beta\gamma,\ \delta\gamma.$$

Observe that these relations, together with $\varepsilon\alpha\delta - \varepsilon\gamma\beta\alpha\delta$ and $\varepsilon\alpha\delta\alpha - \varepsilon\gamma\beta\alpha\delta\alpha$, as well as eight monomial relations of the forms $\beta\gamma q$ and $\delta\gamma q$, generate I as a left ideal. Moreover, consider the mast $p = \varepsilon\gamma\beta\alpha\delta\alpha$. The detours on p are (β,α), (ε,α), and $(\varepsilon,\alpha\delta\alpha)$. Inserting the corresponding substitution equations

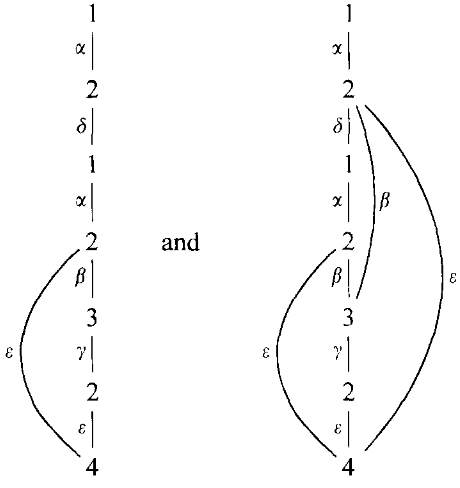
$$\beta\alpha \mathrel{\widehat{=}} X_1\beta\alpha\delta\alpha, \qquad \varepsilon\alpha \mathrel{\widehat{=}} X_2 p, \qquad \varepsilon\alpha\delta\alpha \mathrel{\widehat{=}} X_3 p$$

into the relations yields $V_p = V(X_2 - X_1,\ X_3 - 1) = \{(k_1, k_1, 1) \mid k_1 \in K\} \cong \mathbb{A}^1$.

Note that $w = \delta\alpha$ is the only right subpath of positive length of p which ends in e_1, whence the family Z of variables in the system $S_p(X,Y,Z)$ is reduced to a single one. Using the above method for determining the system $S_p(X,Y,Z)$, we obtain, for any two points $k = (k_1, k_1, 1)$ and $k' = (k_1', k_1', 1)$ in V_p:

$$(S_p(k,k',Z)) \qquad k_1' + Z = k_1, \quad k_1' + Z = k_1, \quad 1 = 1.$$

Since this system is consistent for arbitrary choice of $k, k' \in V_p$, there is, up to isomorphism, precisely one uniserial left Λ-module with mast p. The following are all its graphs relative to suitable top elements:



We conclude the section with a look at hereditary algebras. As is readily seen, in that case, all of the varieties V_p are full affine spaces of 'maximum' dimension. We will see that the split hereditary algebras are actually characterized by their varieties of uniserial modules, which answers a question of K.R. Fuller.

Proposition C *The algebra Λ is hereditary if and only if, for each path p in $K\Gamma$, the variety V_p is isomorphic to the full affine space $\mathbb{A}^{N(p)}$, where*

$$N(p) = \sum\{|I(\alpha,u)| \ : \ (\alpha,u) \wr\wr p\}$$

and $\mathbb{A}^0$ stands for a singleton.

In case these conditions are satisfied, all the maps

$$\Phi_p : V_p \to \{\text{isomorphism types of uniserials in } \Lambda\text{-mod with mast } p\}$$

are bijections, i.e., $\Phi_p(k) \cong \Phi_p(k')$ precisely when $k = k'$. Moreover, for a hereditary algebra Λ over an infinite field K, the following statements are equivalent:

(i) *There are only finitely many isomorphism types of uniserial Λ-modules.*

(ii) *Given any pair of vertices e and e' in Γ and an arrow $\alpha : e \to e'$, the arrow α is the only* path *from e to e' in $K\Gamma$.*

(iii) *For any finite sequence of simple Λ-modules there is either no or precisely one uniserial module having this composition series, depending on whether or not there is a path in $K\Gamma$ which passes through the corresponding sequence of vertices.*

Proof. To prove the first equivalence, observe that, for any path p, the coordinate ring of the variety V_p is $K[X_i(\alpha,u) \mid i \in I(\alpha,u),\ (\alpha,u) \wr\wr p]$ modulo the ideal generated by all the polynomials arising from an insertion of the substitution equations into relations involving routes on p. In particular, this ideal will be nonzero whenever p makes a nontrivial appearance in a relation of Λ, in which case $\dim V_p < N(p)$. Consequently, isomorphism of V_p with $\mathbb{A}^{N(p)}$ for all paths p does not allow for any nontrivial relations. Conversely, it is clear that $V_p \cong \mathbb{A}^{N(p)}$ for all p when Λ is hereditary.

Now suppose that Λ is hereditary. Since the quiver Γ is acyclic in this case, bijectivity of the maps Φ_p follows from part (III) of Theorem A. In view of the first part of the theorem, condition (i) is therefore equivalent to '$N(p) = 0$' for all p, whence (i) implies (iii), the converse of this implication being trivial. But condition (ii) also translates into the nonexistence of detours on any path p in $K\Gamma$, that is, into the equality '$N(p) = 0$' for all p. This completes the proof. □

5. Varieties of uniserials with fixed sequence of composition factors

A more natural subdivision of the varieties of uniserial modules than that in terms of masts – the latter depending a priori on the given coordinatization of Λ – is in terms of sequences of consecutive composition factors. In order to understand the uniserials of composition length $l+1$ with a fixed sequence $(S(1),\dots,S(l+1))$ of simple composition factors, we need to first study the correlation among the varieties V_p, where p runs through all paths of length l which pass precisely through the vertices $e(1),\dots,e(l+1)$, in that order. In particular, we need to explore the intersections $\Phi_p(V_p) \cap \Phi_q(V_q)$, where p and q are two such paths. In a first easy step, we will observe that there is a

1–1 correspondence between the detours on p and those on q which preserves the cardinalities of the corresponding index sets $I(\alpha,u)$. Hence, the varieties V_p and V_q live in the same affine space $\mathbb{A}^N$. Let $D = \Phi_p(V_p) \cap \Phi_q(V_q) = \operatorname{Im}(\Phi_p) \cap \operatorname{Im}(\Phi_q)$. It turns out that the preimages $\Phi_p^{-1}(D)$ and $\Phi_q^{-1}(D)$ are Zariski-open in V_p and V_q, respectively, and isomorphic. (We do not require varieties to be irreducible and correspondingly mean by an isomorphism between two varieties a homeomorphism which has regular coordinate functions in both directions.) In particular, the irreducible components of V_p which intersect V_q and those of V_q which intersect V_p can be paired off into pairs of birationally equivalent partners. (Recall that, by Definition 8, an irreducible component W of V_p intersects V_q if $\Phi_p(W) \cap \operatorname{Im}(\Phi_q) \neq \emptyset$.) The gist of this is that, if we are looking for a set of representatives of the birational equivalence classes of all the uniserial varieties of Λ-mod at the paths p running through the above sequence of vertices, we will not lose information in proceeding as follows: Let $q_1, \dots, q_t$ be the distinct paths of length l passing through the sequence $(e(1), \dots, e(l+1))$. We start by determining the irreducible components of V_{q_1}. Then we find the irreducible components W of V_{q_2} such that $\Phi_{q_2}(W) \cap \operatorname{Im}(\Phi_{q_1}) = \emptyset$, next the components W of V_{q_3} with the property that $\Phi_{q_3}(W) \cap (\operatorname{Im}(\Phi_{q_1}) \cup \operatorname{Im}(\Phi_{q_2})) = \emptyset$, and so forth. Eventually, this procedure will lead us to a family of irreducible affine varieties which contains a representative of each birational equivalence class occurring among the irreducible components of the V_q, $q = q_1, \dots, q_t$.

Lemma 11. *Suppose that p and q are paths of length l passing through the same sequence of vertices $(e(1), \dots, e(l+1))$ in the given order. Then there is a bijection ρ from the set of detours on p to the set of detours on q such that $|I(\alpha,u)| = |I(\rho(\alpha,u))|$ for all detours $(\alpha,u) \wr p$. In particular, if*

$$N = \sum_{(\alpha,u)\wr p} |I(\alpha,u)|,$$

then V_p and V_q are both subvarieties of affine N-space $\mathbb{A}^N$ over K.

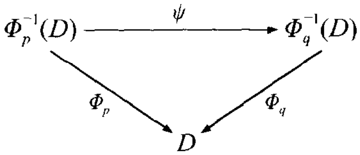
Proof. It clearly suffices to focus on the case where p and q differ in precisely one arrow; an obvious induction will then complete the proof. Say $p = \alpha_l \cdots \alpha_1$ and $q = \alpha_l \cdots \alpha_{r+1}\beta_r\alpha_{r-1} \cdots \alpha_1$. Let ρ be the identity on those detours (γ,u) on p for which either length $u < r-1$, or else length $u = r-1$ and $\gamma \neq \beta_r$, and set $\rho(\beta_r, \alpha_{r-1} \cdots \alpha_1) = (\alpha_r, \alpha_{r-1} \cdots \alpha_1)$ if $\gamma = \beta_r$. Any detour of the form $(\gamma, \alpha_s \cdots \alpha_1)$ on p with $s \geq r$, finally, we match up with the detour $(\gamma, \alpha_s \cdots \alpha_{r+1}\beta_r\alpha_{r-1} \cdots \alpha_1)$ on q. It is easy to check that this bijection ρ preserves the cardinalities of the corresponding index sets as claimed. □

The proof of the following theorem meets with a few technical hurdles which will, however, cease to play a role in the further development of the subject.

Theorem D *Let p and q be paths of length l passing through the same sequence of vertices, and let N be as in the preceding lemma. Moreover, consider the intersection*

$D = \Phi_p(V_p) \cap \Phi_q(V_q)$. Then the preimages $\Phi_p^{-1}(D)$ and $\Phi_q^{-1}(D)$ in $\mathbb{A}^N$ are isomorphic Zariski open subsets of V_p and V_q, respectively.

More precisely, if p and q differ in exactly s arrows, then there exist Zariski open subsets Z_1 and Z_2 of the form $Z_i = \mathbb{A}^N \setminus V(X_{i1} \cdots X_{is})$ in $\mathbb{A}^N$ such that $Z_1 \cap V_p = \Phi_p^{-1}(D)$ and $Z_2 \cap V_q = \Phi_q^{-1}(D)$, together with an isomorphism $\psi: Z_1 \cap V_p \to Z_2 \cap V_q$ which makes the following diagram commutative:



Proof. To construct an isomorphism $\psi: \Phi_p^{-1}(D) \to \Phi_q^{-1}(D)$, we assume that $D \neq \emptyset$. We start with a point $k \in \Phi_p^{-1}(D)$, and set $U = \Phi_p(k)$. Moreover, we let $x \in U$ be a top element such that, for each detour (γ, u) on p, we have $\gamma u x = \sum_i k_i(\gamma, u) v_i(\gamma, u) x$. Then k is the family of coordinate vectors $(k_i(\gamma, u))_{i \in I(\gamma, u)}$ of the elements $\gamma u x$ with respect to the basis $(v_i(\gamma, u)x)$, $i \in I(\gamma, u)$, for the K-space $U(\gamma, u) = eJ^{\text{length}(u)+1}U$, where e is the endpoint of γ. Let $\rho(\gamma, u) = (\gamma', u')$ where ρ is as in Lemma 11; the lemma then allows us to assume that $I(\gamma', u') = I(\gamma, u)$. As we will see, the coordinate vector $(k'_i(\gamma', u'))_{i \in I(\gamma', u')}$ of the element $\gamma' u' x$ relative to the basis $v'_i(\gamma', u')x$, $i \in I(\gamma', u')$, for $U(\gamma, u) = U(\gamma', u')$ does not depend on the choice of the element x as above. The assignment $k \mapsto k' = (k'_i(\gamma', u'))_{i \in I(\gamma', u'), (\gamma', u') \wr q}$ thus yields a well-defined map $\psi: \Phi_p^{-1}(D) \to \Phi_q^{-1}(D)$ by the Corollary to Theorem A. Furthermore, we will find that the coordinates of ψ are rational functions in the $k_i(\gamma, u)$ which are defined on the whole domain and which depend only on p and q. In particular, this will show that the coordinate functions of ψ are regular maps. Once this is established, it will readily follow that ψ is an isomorphism from $\Phi_p^{-1}(D)$ to $\Phi_q^{-1}(D)$. Indeed, if $\psi': \Phi_q^{-1}(D) \to \Phi_p^{-1}(D)$ is the map constructed in complete analogy to ψ, then ψ' has regular coordinate functions by symmetry, and it is straightforward to check that $\psi'\psi$ and $\psi\psi'$ are the identities on $\Phi_p^{-1}(D)$ and $\Phi_q^{-1}(D)$, respectively.

As in the proof of Lemma 11, we will assume that p and q differ in precisely one arrow and leave the general case to the reader. Say

$$p = \alpha_l \cdots \alpha_1 \quad \text{and} \quad q = \alpha_l \cdots \alpha_{r+1} \beta_r \alpha_{r-1} \cdots \alpha_1,$$

where each α_i is an arrow $e(i) \to e(i+1)$ and β_r is an arrow $e(r) \to e(r+1)$. Set $u_0 = \alpha_{r-1} \cdots \alpha_1$. Clearly, we then have $\Phi_p^{-1}(D) = \{k \in V_p \mid k_1(\beta_r, u_0) \neq 0\}$ and $\Phi_q^{-1}(D) = \{k' \in V_q \mid k'_1(\alpha_r, u_0) \neq 0\}$, where we have chosen our indices so that $v_1(\beta_r, u_0) = \alpha_r u_0$ and $v'_1(\alpha_r, u_0) = \beta_r u_0$. In other words, if $Z_1 = \mathbb{A}^N \setminus V(X_1(\beta_r, u_0))$ and $Z_2 = \mathbb{A}^N \setminus V(X_1(\alpha_r, u_0))$, then $\Phi_p^{-1}(D) = Z_1 \cap V_p$ and $\Phi_q^{-1}(D) = Z_2 \cap V_q$.

To follow the strategy outlined above, we let (γ, u) be a detour on p, and treat the following cases separately:

1. $u = u_0$ and $\gamma = \beta_r$. Then $(\gamma', u') = (\alpha_r, u_0)$.
2. $u = u_0$ and $\gamma \neq \beta_r$. Then $(\gamma', u') = (\gamma, u)$.
3. length$(u) <$ length$(u_0) = r - 1$. Then $(\gamma', u') = (\gamma, u)$.
4. length$(u) >$ length(u_0). If $u = \alpha_s \cdots \alpha_1$, then $(\gamma', u') = (\gamma, \alpha_s \cdots \alpha_{r+1}\beta_r u_0)$.

Throughout, assume that $I(\gamma, u)$ is a set of natural numbers and that the paths $v_i(\gamma, u)$ and $v_i'(\gamma', u')$ are ordered by length, that is,

$$\text{length}\, v_i(\gamma, u) < \text{length}\, v_{i+1}(\gamma, u)$$

for all i, and similarly for (γ', u'); then, clearly, length $v_i(\gamma, u)$ = length $v_i'(\gamma', u')$ for all $i \in I(\gamma, u) = I(\gamma', u')$. Again, let $k \in \Phi_p^{-1}(D) = Z_1 \cap V_p$, and identify $\Phi_p(k)$ with a uniserial module U having top element x such that $\delta v x = \sum k_i(\delta, v) v_i(\delta, v) x$ for each detour (δ, v) on p.

In case 1, we write each path $v_i'(\alpha_r, u_0)$ in the form $u_i \beta_r u_0$, where u_i is a path of length ≥ 0; in particular, since $v_1'(\alpha_r, u_0) = \beta_r u_0$, we have $u_1 = e(r+1)$. We infer that $v_i(\beta_r, u_0) = u_i \alpha_r u_0 = u_i v_1(\beta_r, u_0)$ and compute

$$\begin{aligned}
\alpha_r u_0 x &= \sum_{i \in I(\alpha_r, u_0)} k_i'(\alpha_r, u_0) v_i'(\alpha_r, u_0) x \\
&= \sum_{i \in I(\alpha_r, u_0)} k_i'(\alpha_r, u_0) u_i \left(\sum_{j \in I(\beta_r, u_0)} k_j(\beta_r, u_0) v_j(\beta_r, u_0) x \right) \\
&= \sum_{i \geq 1} k_i'(\alpha_r, u_0) k_1(\beta_r, u_0) v_i(\beta_r, u_0) x + \sum_{\substack{i \geq 1 \\ j \geq 2}} k_i'(\alpha_r, u_0) k_j(\beta_r, u_0) u_i v_j(\beta_r, u_0) x.
\end{aligned}$$

Observing that, for $i \geq 1$ and $j \geq 2$, we have length$(u_i v_j(\beta_r, u_0)) >$ length$(u_i v_1(\beta_r, u_0))$ = length$(v_i(\beta_r, u_0))$ and length$(u_i v_j(\beta_r, u_0)) \geq$ length$(v_j(\beta_r, u_0))$, we obtain

$$u_i v_j(\beta_r, u_0) x = \sum_{s \geq \max(i+1, j)} \sigma_{ijs} v_s(\beta_r, u_0) x$$

for all $i \geq 1$ and $j \geq 2$, where the σ_{ijs} are polynomials in the $k_t(\delta, v)$, for detours (δ, v) on p, which – aside from dependence on the indices – depend only on p, q and the detour considered.

Inserting the second equality into the first yields

$$\alpha_r u_0 x = \sum_{i \geq 1} \left(k_i'(\alpha_r, u_0) k_1(\beta_r, u_0) + \sum_{\substack{1 \leq s \leq i-1 \\ 2 \leq j \leq i}} k_s'(\alpha_r, u_0) k_j(\beta_r, u_0) \sigma_{sji} \right) v_i(\beta_r, u_0) x.$$

On the other hand, $\alpha_r u_0 x = 1 \cdot v_1(\beta_r, u_0) x$, and since the elements $v_i(\beta_r, u_0) x$ of U are K-linearly independent, a comparison of coefficients leads us to the system of equations

$k_1'(\alpha_r, u_0)k_1(\beta_r, u_0) = 1$ and

$$k_i'(\alpha_r, u_0)k_1(\beta_r, u_0) + \sum_{1 \le s \le i-1} k_s'(\alpha_r, u_0) \left(\sum_{2 \le j \le i} k_j(\beta_r, u_0)\sigma_{sji} \right) = 0$$

for $i \ge 2$. This system for the 'unknowns' $k_i'(\alpha_r, u_0)$ has size $|I(\beta_r, u_0)| \times |I(\beta_r, u_0)|$ and is lower triangular, the scalar $k_1(\beta_r, u_0)$ holding all positions along the main diagonal of the coefficient matrix; this scalar is nonzero, because we chose $k \in Z_1$. Consequently, the system is uniquely solvable for the $k_i'(\alpha_r, u_0)$, and the solution is of the form $k_i'(\alpha_r, u_0) = \tau_i/(k_1(\beta_r, u_0))^i$, where the τ_i are polynomials in the $k_t(\delta, v)$ depending only on i, p, q and on the detour (α_r, u_0) on q.

We will leave the details of cases 2, 3 to the reader, but will carry out case 4. In that case, length $v_i'(\gamma', u') =$ length $v_i(\gamma, u) >$ length$(u') =$ length$(u) >$ length(u_0), and hence $v_i'(\gamma', u') = u_i\beta_r u_0$ for some path u_i of length ≥ 1, while $v_i(\gamma, u) = u_i\alpha_r u_0$ for all $i \in I(\gamma', u') = I(\gamma, u)$.

Again we compute $\gamma' u' x$ in two ways. On one hand,

$$\begin{aligned}
\gamma' u' x &= \sum_{i \in I(\gamma', u')} k_i'(\gamma', u') u_i \beta_r u_0 x \\
&= \sum_{i \in I(\gamma, u)} k_i'(\gamma', u') u_i \left(\sum_{j \in I(\beta_r, u_0)} k_j(\beta_r, u_0) v_j(\beta_r, u_0) x \right) \\
&= \sum_{i \in I(\gamma, u)} k_i'(\gamma', u') k_1(\beta_r, u_0) v_i(\gamma, u) x \\
&\quad + \sum_{\substack{i \in I(\gamma, u) \\ j \in I(\beta_r, u_0), j \ge 2}} k_i'(\gamma', u') k_j(\beta_r, u_0) u_i v_j(\beta_r, u_0) x.
\end{aligned}$$

Here we use the facts that $I(\gamma', u') = I(\gamma, u)$ and that $u_i v_1(\beta_r, u_0) = u_i \alpha_r u_0 = v_i(\gamma, u)$.

To evaluate the terms $u_i v_j(\beta_r, u_0) x$ for $j \ge 2$, we observe that

$$\text{length}(u_i v_j(\beta_r, u_0)) > \text{length}(u_i v_1(\beta_r, u_0)) = \text{length}(u_i \alpha_r u_0) = \text{length}(v_i(\gamma, u))$$

for all $j \ge 2$. Thus, for $j \ge 2$, we have $u_i v_j(\beta_r, u_0) x = \sum_{s \ge i+1} \sigma_{ijs} v_s(\gamma, u) x$, where the σ_{ijs} are again polynomials in the $k_t(\delta, v)$ depending solely on the indices, the paths p, q, and the detour (γ, u). Inserting this information into the above equality yields

$$\text{(I)} \qquad \gamma' u' x = \sum_{i \in I(\gamma, u)} \Bigg(k_i'(\gamma', u') k_1(\beta_r, u_0) + \sum_{\substack{1 \le s \le i-1 \\ s \in I(\gamma, u)}} k_s'(\gamma', u') \sum_{\substack{j \in I(\beta_r, u_0) \\ j \ge 2}} k_j(\beta_r, u_0) \sigma_{sji} \Bigg) v_i(\gamma, u) x.$$

On the other hand, we can write $u = w\alpha_r u_0$ for a suitable path w of length ≥ 0. Then $u' = w\beta_r u_0$, and we obtain

$$\gamma' u' x = \gamma w \beta_r u_0 x = \sum_{j \in I(\beta_r, u_0)} k_j(\beta_r, u_0) \gamma w v_j(\beta_r, u_0) x.$$

Now each of the paths $\gamma w v_j(\beta_r, u_0)$ is longer than u and has the same endpoint as γ. Therefore

$$\gamma w v_j(\beta_r, u_0) x = \sum_{i \in I(\gamma, u)} \tau_{ji} v_i(\gamma, u) x,$$

where the $v_i(\gamma, u)$ are suitable polynomials in the $k_t(\delta, v)$. This gives us

$$\text{(II)} \qquad \gamma' u' x = \sum_{i \in I(\gamma, u)} \left(\sum_{j \in I(\beta_r, u_0)} k_j(\beta_r, u_0) \tau_{ji} \right) v_i(\gamma, u) x,$$

and a comparison of coefficients of the K-linearly independent elements $v_i(\gamma, u)x \in U$ in equations (I), (II) once more yields a square system of equations for the $k_i'(\gamma', u')$. Again the coefficient matrix is lower triangular, and all diagonal positions are occupied by $k_1(\beta_r, u_0)$. Since $k_1(\beta_r, u_0) \neq 0$ by the choice of k, the system has a unique solution, expressing the $k_i'(\gamma', u')$ as polynomials in the $k_t(\delta, v)$ divided by powers of $k_1(\beta_r, u_0)$. □

Let $W_1, \ldots, W_m$ be a full set of representatives for the birational equivalence classes of *all* the irreducible components occurring in the varieties V_p, where p runs through the sequence of vertices $(e(1), \ldots, e(l+1))$. Since, in particular, W_i is not birationally equivalent to W_j for $i \neq j$, Theorem D tells us that, for any choice of $i \neq j$ and $p \neq q$ with $W_i \subseteq V_p$ and $W_j \subseteq V_q$, the intersection $\Phi_p(W_i) \cap \Phi_q(W_j)$ is empty; in other words, among the irreducible varieties W_i selected above, any two corresponding to different masts are disjoint. We can therefore determine the set $\{W_1, \ldots, W_m\}$ in the manner described in the beginning of the section. In particular, Theorem D guarantees that the result will not depend on the ordering of the paths $p_1, \ldots, p_t$; in other words, the set of birational equivalence classes obtained will be invariant under permutation of the p_i.

We interrupt the theory to give two examples. In each case, p and q are two paths of the same length passing through the same sequence of vertices. In the first example, V_p and V_q are irreducible with $\Phi_p(V_p) \cap \Phi_q(V_q) \neq \emptyset$, but $V_p \not\cong V_q$; Theorem D guarantees that V_p and V_q are birationally equivalent in that case. In the second, again $\mathrm{Im}(V_p) \cap \mathrm{Im}(V_q) \neq \emptyset$, but this time V_p is irreducible while V_q is not; in this situation, Theorem D guarantees that V_p is birationally equivalent to an irreducible component of V_q.

Example 12. Let Γ be the quiver

$$1 \underset{\alpha}{\overset{\alpha'}{\rightrightarrows}} 2 \underset{\beta}{\overset{\beta'}{\rightrightarrows}} 3$$

and $\Lambda = K\Gamma/I$, where I is generated by $\beta'\alpha' - \beta\alpha$. If p is equal to $\beta\alpha$ or $\beta'\alpha'$ and q to $\beta'\alpha$ or $\beta\alpha'$, then $V_p = V(X_1X_2 - 1)$ is isomorphic to the punctured line, while $V_q = V(X_1 - X_2)$ is isomorphic to the full affine line over K. Thus $V_p \not\cong V_q$. Note, however, that $\Phi_p(V_p) \cap \Phi_q(V_q) \neq \emptyset$.

Example 13. Let Γ be the quiver

$$1 \underset{\alpha}{\overset{\alpha'}{\rightrightarrows}} 2 \underset{\beta}{\overset{\beta'}{\rightrightarrows}} 3 \underset{\gamma}{\overset{\gamma'}{\rightrightarrows}} 4 \underset{\delta}{\overset{\delta'}{\rightrightarrows}} 5 \underset{\varepsilon}{\overset{\varepsilon'}{\rightrightarrows}} 6$$

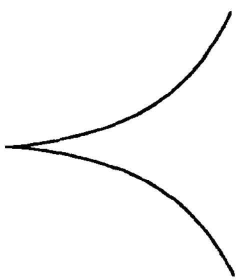
and $\Lambda = K\Gamma/I$, where I is the ideal generated by the relations

$$\varepsilon'\delta'\gamma\beta\alpha - \varepsilon\delta\gamma'\beta'\alpha', \qquad \varepsilon\delta\gamma\beta'\alpha - \varepsilon\delta\gamma\beta\alpha',$$
$$\varepsilon\delta\gamma'\beta\alpha - \varepsilon\delta\gamma\beta\alpha', \qquad \varepsilon\delta'\gamma\beta\alpha - \varepsilon'\delta\gamma\beta\alpha.$$

If $p = \varepsilon\delta\gamma\beta\alpha$ and $q = \varepsilon'\delta\gamma\beta'\alpha$, then

$$V_p = V(X_4X_5 - X_1X_2X_3,\ X_2 - X_1,\ X_3 - X_1,\ X_4 - X_5) \cong V(Y^2 - X^3)$$

is the standard cusp:



In particular, V_p is irreducible. On the other hand,

$$\begin{aligned} V_q &= V(X_2, X_5) \cup V(X_4X_2 - X_5X_3X_1,\ X_1X_2 - 1,\ X_3 - X_1,\ X_5X_4 - 1) \\ &\cong \mathbb{A}^3 \cup V(X_4^2 - X_1^3,\ X_1X_2 - 1,\ X_5X_4 - 1), \end{aligned}$$

the latter component being isomorphic to the cusp from which the singular point has been deleted, or, in other words, isomorphic to the punctured line. Note that, in this case, still $\Phi_p(V_p) \cap \Phi_q(V_q) \neq \emptyset$, but $\Phi_p(V_p) \cap \Phi_q(V(X_2, X_5)) = \emptyset$.

As repeatedly announced, the birational equivalence classes of the irreducible components of the varieties V_p, where p runs through the paths of length l through a fixed sequence $(e(1), \ldots, e(l+1))$ of vertices, form an isomorphism invariant of the algebra Λ. To see this, we will require a technical theorem which describes the behavior of the varieties V_p under change of coordinatization. The proof of this theorem constitutes a rather lengthy detour from our main train of thought; in particular, it is based on an extension of our conceptual framework. Therefore, it will be carried out separately in [5]. The statement of the theorem is as follows:

Theorem E. *Given are two coordinatizations of Λ, namely $\Lambda = K\Gamma/I \cong K\widehat{\Gamma}/\widehat{I}$, where the quiver Γ is based on the primitive idempotents $e_1, \ldots, e_n$ as before, and $\widehat{\Gamma}$ (necessarily isomorphic to Γ as a directed graph) is based on primitive idempotents $\widehat{e}_1, \ldots, \widehat{e}_n$ of Λ, ordered in such a way that each e_i is congruent, modulo J, to the image of $\widehat{e}_i$ under our isomorphism.*

If $\widehat{p}$ is a path of length l in $K\widehat{\Gamma}$ passing through the sequence of vertices $(\widehat{e}(1), \ldots, \widehat{e}(l+1))$, then each irreducible component of $V_{\widehat{p}}$ is birationally equivalent to a component of some variety V_p, where p is a path of length l in $K\Gamma$ passing through the sequence $(e(1), \ldots, e(l+1))$. Here $V_{\widehat{p}}$ is the variety of $\widehat{p}$ relative to the coordinates $\widehat{\Gamma}$ and $\widehat{I}$, and V_p the variety of p relative to Γ and I.

Theorem F and Definition. *Fix a sequence $\mathbb{S} = (S(1), \ldots, S(l+1))$ of simple left Λ-modules and let $(e(1), \ldots, e(l+1))$ be the corresponding canonical sequence of primitive idempotents in a fixed coordinatization $\Lambda = K\Gamma/I$. Then the set of birational equivalence classes of the irreducible components of the nonempty varieties V_p, where p runs through the paths of length l in $K\Gamma$ passing through $(e(1), \ldots, e(l+1))$ in that order, is independent of the coordinatization. In other words, the set $V_{\mathbb{S}}$ consisting of the irreducible components of the nonempty varieties V_p as above is uniquely determined by the isomorphism type of Λ, up to birational equivalence. We denote by $V_{\mathbb{S}}$ the set of* irreducible uniserial varieties of Λ at $\mathbb{S}$.

Moreover, if $V_{\mathbb{S}} = \{W_1, \ldots, W_m\}$, the uniserial genus of Λ at $\mathbb{S}$ *is defined to be the maximum of the numbers* $\text{genus}(W_1), \ldots, \text{genus}(W_m)$, *and is denoted by* $\text{genus}\, V_{\mathbb{S}}$. *By the preceding paragraph,* $\text{genus}\, V_{\mathbb{S}}$ *is uniquely determined by the isomorphism class of Λ.*

An example of nontrivial uniserial genus can be found in Section 6 (Example 14).

6. Realization of arbitrary varieties as varieties of uniserials

The aim of this section is to show that each affine variety V is isomorphic to a variety of uniserial modules over a suitable finite dimensional algebra $\Lambda = K\Gamma/I$, under the additional restrictions that Γ be acyclic and without double arrows. More precisely, we will realize our given variety V as a variety V_p for some path p in Γ. Let $\mathbb{S}$ be the sequence of simple modules corresponding to the consecutive vertices along p. Observe that, due to the absence of double arrows, $V_{\mathbb{S}}$ then consists precisely of the irreducible components of V_p. In other words, the set of irreducible components of an arbitrary variety V can be realized as a $V_{\mathbb{S}}$.

By Theorem A(III), the first of the above restrictions on Γ guarantees that Φ_p is bijective, meaning that there is a 1–1 correspondence between the points of V_p and the uniserial left Λ-modules with mast p. The second condition on Γ ensures that all of the varieties V_q where q runs through the paths in $K\Gamma$, are actually isomorphism invariants of Λ [5]. In particular, our theorem thus provides a taste of the opulence and complexity of uniserial representations of finite dimensional algebras.

Theorem G. *Given a field K and any affine algebraic variety V over K, there exists a finite dimensional path algebra modulo relations, $\Lambda = K\Gamma/I$, together with a path p in $K\Gamma$, such that $V \cong V_p$. We can, moreover, choose the quiver Γ to be acyclic and without double arrows.*

Proof. Say $V = V(f_1, \dots, f_M)$, where the f_i are polynomials in $K[X_1, \dots, X_m]$ for some $m \geq 1$. In a preliminary step, we construct an affine variety V' isomorphic to V such that V' is the vanishing set of polynomials f' in a certain polynomial ring $K[X_{ij}]$ which have the property that none of the variables X_{ij} occurs in a power higher than 1 in any monomial.

To that end, let $d_i = \max_{1 \leq j \leq M} \deg_{X_i}(f_j)$ for $1 \leq i \leq m$, and introduce new variables $X_{11}, X_{12}, \dots, X_{1d_1}, X_{21}, X_{22}, \dots, X_{2d_2}, \dots, X_{m1}, \dots, X_{md_m}$. For each $s \in \{1, \dots, M\}$, construct a polynomial $f'_s \in K[X_{ij} \mid 1 \leq i \leq m,\ 1 \leq j \leq d_i]$ as follows: Replace any monomial $X_1^{r_1} \cdots X_m^{r_m}$ occurring nontrivially in f_s by

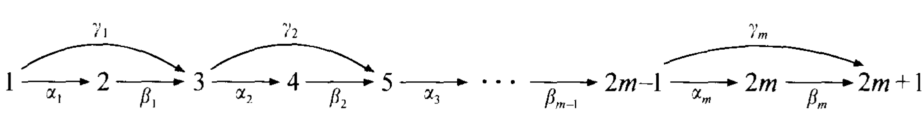
$$(X_{11} \cdots X_{1r_1})(X_{21} \cdots X_{2r_2}) \cdots (X_{m1} \cdots X_{mr_m})$$

in $K[X_{ij}]$; this is possible, since $r_i \leq d_i$ for $1 \leq i \leq m$ by construction. Clearly, the total degree of this new monomial in the X_{ij} is the same as that of $X_1^{r_1} \cdots X_m^{r_m}$. Define

$$V' = V(f'_1, \dots, f'_M,\ X_{is} - X_{it} \mid 1 \leq i \leq m,\ 1 \leq s, t \leq d_i).$$

That $V \cong V'$ is then an obvious consequence of our construction, and hence we may assume, without loss of generality, that $V = V'$; in other words, we may assume that $d_i \leq 1$ for all $i \in \{1, \dots, m\}$. This concludes the preliminary step.

We now let Γ be the quiver



and set $p = \beta_m \alpha_m \cdots \beta_2 \alpha_2 \beta_1 \alpha_1$. To define suitable relations in $K\Gamma$, we write each f_j in the form $f_j = \sum_{A \in \mathscr{P}} c_j(A) \prod_{i \in A} X_i$, where $\mathscr{P}$ is the power set of $\{1, \dots, m\}$ and $c_j(A) \in K$; this is possible by the preceding paragraph. For each $A \in \mathscr{P}$, define a path

$p(A)$ in $K\Gamma$ as $p(A) = p_m(A) \cdots p_1(A)$, where the $p_i(A)$ are defined by

$$p_i(A) = \begin{cases} \gamma_i & \text{if } i \in A, \\ \beta_i \alpha_i & \text{if } i \notin A. \end{cases}$$

Note that each $p(A)$ is a path from 1 to $2m+1$. Let $r_j = \sum_{A \in \mathscr{P}} c_j(A) p(A)$ for $1 \le j \le M$, and observe that each r_j is a relation in $K\Gamma$. Finally, define $\Lambda = K\Gamma/I$, where $I \subseteq K\Gamma$ is the ideal generated by $r_1, \ldots, r_M$, and note that $I^{(L)} = I^{(2m+1)} = I$ is generated by the r_j as a K-space. We now verify that $V \cong V_p$ as follows.

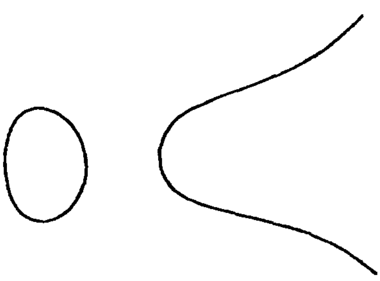
Clearly, there are no paths in $K\Gamma$ starting in 1 which fail to be routes on p, and the detours on p are precisely the pairs $(\gamma_i, \beta_{i-1}\alpha_{i-1} \cdots \beta_1\alpha_1)$ for $1 \le i \le m$; here $\beta_0\alpha_0$ stands for the primitive idempotent e_1 identified with the vertex 1. Moreover, for each detour (γ, u) on p, there exists precisely one right subpath $v(\gamma, u) = v_1(\gamma, u)$ of p longer than u and ending in the same vertex as γ, namely $v(\gamma_i, \beta_{i-1}\alpha_{i-1} \cdots \beta_1\alpha_1) = \beta_i\alpha_i \cdots \beta_1\alpha_1$. Writing X_i for the variable $X_1(\gamma_i, \beta_{i-1}\alpha_{i-1} \cdots \beta_1\alpha_1)$, we thus obtain the substitution equations

$$\begin{aligned} \gamma_1 e_1 &\mathrel{\widehat{=}} X_1 \beta_1 \alpha_1, \\ \gamma_2 \beta_1 \alpha_1 &\mathrel{\widehat{=}} X_2 \beta_2 \alpha_2 \beta_1 \alpha_1, \\ &\vdots \\ \gamma_m \beta_{m-1} \alpha_{m-1} \cdots \beta_1 \alpha_1 &\mathrel{\widehat{=}} X_m \beta_m \alpha_m \cdots \beta_1 \alpha_1 = X_m p. \end{aligned}$$

Inserting these successively into the relations r_j from the right yields, after at most m steps, the equations $r_j \mathrel{\widehat{=}} f_j(X_1, \ldots, X_m) p$, this being an immediate consequence of our construction. Therefore $V_p = V(f_1, \ldots, f_M) = V$ as desired. □

Note that the proof of Theorem G is constructive. We conclude by applying the pertinent algorithm to obtain an algebra with a uniserial variety of positive genus.

Example 14. Let $V = V(X_2^2 - X_1(X_1^2 - 1))$ be the elliptic curve with $\mathbb{R}$-graph



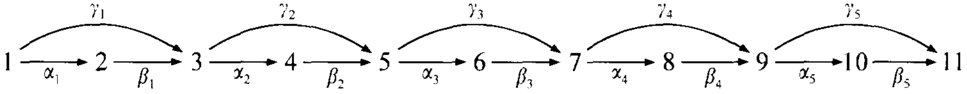
Introduce the new variables $X_{11}, X_{12}, X_{13}, X_{21}, X_{22}$ and note that the variety

$$V' = V(X_{21}X_{22} - X_{11}X_{12}X_{13} + X_{11},\ X_{11} - X_{12},\ X_{11} - X_{13},\ X_{21} - X_{22})$$

is isomorphic to V. The following change of indexing makes it easier to follow the pattern described in the proof of Theorem F:

$$V' = V(X_5X_4 - X_3X_2X_1 + X_1,\ X_1 - X_2,\ X_1 - X_3,\ X_4 - X_5).$$

Accordingly, we consider the quiver



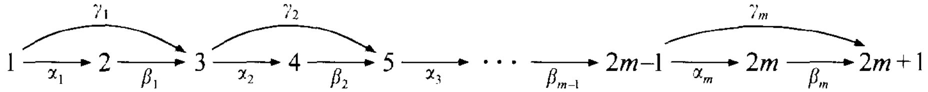
Let $p = q_5q_4q_3q_2q_1$, where $q_i = \beta_i\alpha_i$ for $1 \leq i \leq 5$, and define $\Lambda = K\Gamma/I$, where I is the ideal generated by the following relations:

$$\gamma_5\gamma_4q_3q_2q_1 - q_5q_4\gamma_3\gamma_2\gamma_1 + q_5q_4q_3q_2\gamma_1, \qquad q_5q_4q_3q_2\gamma_1 - q_5q_4q_3\gamma_2q_1,$$

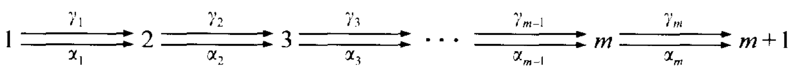
$$q_5q_4q_3q_2\gamma_1 - q_5q_4\gamma_3q_2q_1, \qquad q_5\gamma_4q_3q_2q_1 - \gamma_5q_4q_3q_2q_1.$$

Then $V_p \cong V'$, as substantiated in the proof of the theorem. In particular, if $\mathbb{S} = (S(1), \ldots, S(11))$, we obtain genus $V_{\mathbb{S}} = 1$.

Remark. Note that, given an affine variety V, the K-dimension of the algebra Λ over which V is realized as a variety of uniserial modules increases steeply as the number of variables and their exponents in a description of V grow. Indeed, if $\Gamma(m)$ is the quiver



then $\dim_K K\Gamma(m) = 9 \cdot 2^m - 3m - 8$. If one renounces the requirement that the quiver Γ of the algebra realizing V be without double arrows, one can significantly curb the growth of this dimension in terms of the number of variables involved. Indeed, if one allows for double arrows, the role played by the quiver $\Gamma(m)$ in the proof of Theorem G can be taken over by the quiver $\Gamma'(m)$:



Note that $\dim_K K\Gamma'(m)$ grows less than half as fast as $\dim_K K\Gamma(m)$, but still exponentially.

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Note added in proof. In the meantime, the author and K. Bongartz have proved that the fibres of the maps Φ_p are always isomorphic to full affine spaces $\mathbb{A}^r$ [work in progress].

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